Mathematics 554H Test 1.

In class part of test.

1. What is the sum of the series $S = \sum_{k=2}^{99} 3x^3(1-x)^k$?

Solution: This is a geometric series.

$$\begin{aligned} S &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{3x^3(1-x)^2 - 2x^3(1-x)^{100}}{1-(1-x)} \\ &= \frac{3x^3(1-x)^2 - 2x^3(1-x)^{100}}{x} \\ &= 3x^2(1-x)^2 - 2x^2(1-x)^{100} \end{aligned}$$

□

2.

(a) State the ***binomial theorem***.

Solution: For any positive integer n and any real numbers x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k + y^{n-k}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(b) Simplify $\dfrac{(a+h)^4 - 2a^4 + (a-h)^4}{h^2}$ (the answer should have no h in the denominator).

Solution: Start by expanding the numerator using the $n = 4$ version of the binomial theorem:

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$\begin{aligned} (a+h)^4 - 2a^4 + (a-h)^4 &= a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + a^4 \\ &\quad - 2a^4 \\ &\quad + a^4 - 4a^3h + 6a^2h^2 - 4ah^3 + a^4 \\ &= 12a^2h^2 + 2h^4 \end{aligned}$$

Dividing by h^2 gives the desired result:

$$\frac{(a+h)^4 - 2a^4 + (a-h)^4}{h^2} = 12a^2 + 2h^2.$$

□

3. Give examples of (no proofs required).

(a) A subset of $\mathbb{R}$ that is bounded above, but not bounded below.
Solution: Maybe the most natural examples are $(-\infty, 0)$ or $(-\infty, 0]$. □

(b) A subset of $\mathbb{R}$ that is bounded above, but has no maximum.
Solution: Any open interval (a, b) does the trick. □

(c) An irrational number between 1 and 1.001.
Solution: As $0 < 1/\sqrt{2} < 1$ the number

$$1 + \frac{.001}{\sqrt{2}}$$

is irrational and between 1 and 1.001. □

4. Prove $x^2 + 4xy + 5y^2 \geq 0$ with equality if and only if $x = y = 0$.
Solution: Complete the square:

$$x^2 + 4xy + 5y^2 = x^2 + 4xy + 4y^2 + y^2 = (x + 2y)^2 + y^2 \geq 0$$

is a sum of squares and therefore is always ≥ 0. If this equals zero, then both terms must be zero which gves

$$x + 2y = 0, \qquad y = 0.$$

The second equation gives $y = 0$ and using $y = 0$ in $x + 2y = 0$ gives $x = 0$. □

5. State the four axioms for (E, d) to be a ***metric space***.
Solution: $E \neq \varnothing$ is a nonempty set and $d \colon E \times E \to [0, \infty)$ so that for all $x, y, z \in E$

(i) $d(x, y) \geq 0$.
(ii) $d(x, y) = 0$ if and only if $x = y$.
(iii) $d(x, y) = d(y, x)$
(iv) $d(x, z) \leq d(x, y) + d(y, z)$. □

6.

(a) Define what if means for a function $f \colon [a, b] \to \mathbb{R}$ to be ***Lipschitz***.
Solution: There is a constant $M \geq 0$ so that

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$$

for all $x_1, x_2 \in [a, b]$.

(b) Show the function $f(x) = \dfrac{x+2}{x+3}$ is Lipschitz on the interval $[0, \infty)$.

Solution: Let $a, b \in [0, \infty)$ Then

$$
\begin{aligned}
|f(b) - f(a)| &= \left| \frac{a+2}{a+3} - \frac{b+2}{b+3} \right| \\
&= \left| \frac{(a+2)(b+3) - (a+3)(b+2)}{(a+3)(b+3)} \right| \\
&= \left| \frac{ab + 3a + 2b + 6 - (ab + 2a + 3b + 6)}{(a+3)(b+3)} \right| \\
&= \left| \frac{a-b}{(a+3)(b+3)} \right| \\
&= \frac{|a-b|}{(a+3)(b+3)} \\
&\leq \frac{|a-b|}{(0+3)(0+3)} \\
&= \frac{1}{9}|a-b| \\
&= M|b-a|
\end{aligned}
$$

where $M = 1/9$ we have used that $a, b \geq 0$ so that $\frac{1}{(a+3)(b+3)} \leq \frac{1}{9}$ □

7. Show $|x| \geq \max\{1, 4(|a) + |b|)\}$ implies

$$1 + \frac{a}{x} + \frac{b}{x^2} \geq \frac{3}{4}.$$

Solution: If $|x| \geq 1$, then $1 \leq |x| \leq |x|^2$ and therefore

$$\frac{1}{|x|^2} \leq \frac{1}{|x|}.$$

If also $|x| \geq 4(|a|+|b|)$ (that is if $|x| \geq \max(1, 4(|a|+|b|))$ we have $\dfrac{1}{|x|} \leq \dfrac{1}{4(|a|+|b|)}$ which implies

$$\begin{aligned}
\left|\frac{a}{c} + \frac{b}{x^2}\right| &\leq \frac{|a|}{|x|} + \frac{|b|}{|x|^2} \\
&\leq \frac{|a|}{|x|} + \frac{|b|}{|x|^2} \\
&\leq \frac{|a|}{|x|} + \frac{|b|}{|x|} \\
&= \frac{|a|+|b|}{|x|} \\
&\leq \frac{|a|+|b|}{4(|a|+|b|)} \\
&= \frac{1}{4}.
\end{aligned}$$

Thus

$$1 + \frac{a}{x} + \frac{b}{x^2} \geq 1 - \left|\frac{a}{x} + \frac{b}{x^2}\right| \geq 1 - \frac{1}{4} = \frac{3}{4}.$$

□

8.

(a) State what it means for $S \subseteq \mathbb{R}$ to be bounded above.
Solution: S is bounded above if and only if there is a $b \in \mathbb{R}$ so that $s \leq b$ for all $s \in S$.

(b) If $S \subseteq \mathbb{R}$ define $b = \sup(S)$.
Solution: $b = (a)(S)$, that if b is the least upper bound for S if and only if b is an upper bound for S and if c is any upper bound for S, then $b \leq c$. □

(c) State the ***Least Upper Bound Axiom***.
Any subset of $\mathbb{R}$ which is bounded above has a least upper bound. □

(d) State ***Archimedes' Axiom*** big form.
Solution: For any $x \in \mathbb{R}$ there is a natural number n with $x < n$. □

(e) Use the Least Upper Bound Axiom to prove Archimedes' Axiom.
Solution: Toward a contradiction assume this is false. Then there is an $x \in \mathbb{R}$ with $n \leq x$ for all $n \in \mathbb{N}$. Therefore $\mathbb{N}$ is bounded above and, by the least upper bound axiom, has a least upper bound $b = \sup(\mathbb{N})$. For any natural number n the number $n+1$ is also a natural number and b is an upper bound for the natural

numbers so

$$n + 1 \leq b.$$

This implies

$$n \leq b - 1$$

for all $n \in \mathbb{N}$ so that $b - 1 < b$ is an upper bound for $\mathbb{N}$. This contradicts that b is the *least* upper bound of $\mathbb{N}$. $\square$

9.

(a) State the Intermediate Value Theorem for Lipschitz functions.
Solution: Let $f\colon [a, b] \to \mathbb{R}$ be Lipschitz so that $f(a)$ and $f(b)$ have opposite signs (that is one is positive and one is negative). Then there is a $\xi \in \mathbb{R}$ with $f(\xi) = 0$. $\square$

(b) Prove that the equation $x^3 = \dfrac{10}{1 + x^2}$ has a solution. (You may assume x^3 and $\dfrac{10}{1 + x^2}$ are Lipschitz on bounded intervals.)
Solution: Let $f(x) = x^3 - \dfrac{10}{1 + x^2}$. Then $f(\xi) = 0$ if and only if ξ is a solution to $x^3 = \dfrac{10}{1 + x^2}$. The function f is the sum of two Lipschitz functions and is therefore also Lipschitz. Note

$$f(0) = 0^3 - \frac{10}{1 + 0^2} - 10 < 0.$$

Also

$$f(3) = 3^3 - \frac{10}{1 + 3^2} = 27 - 1 = 26 > 0.$$

Therefore f changes sign on $[0, 3]$ and thus there is a $\xi \in (0, 3)$ with $f(xi) = 0$. This ξ is a solution to our equation. $\square$

Take home part of test.

Definition. A subset U of a metric space is ***open*** if and only if for all $a \in U$ there is an $r > 0$ so that

$$B(a, r) \subseteq U.$$

$\square$

It is important to note that r depends on a and that there is not just one r that works for all $a \in U$. Anther fact is that the empty set $\varnothing$ is an open set. This is an example of a vacuous implication. That is for any $r > 0$ the implication

$$a \in \varnothing \quad \implies \quad B(a, r) \subseteq \varnothing$$

is true as the hypothesis $a \in \varnothing$ is false and an implication $P \implies Q$ is true when P is false.

Proposition 1. *Every open ball $B(a,r)$ is open.*

1. Prove this. *Hint:* If $b \in B(a,r)$, then $d(a,b) < r$. Let $\rho = r - d(a,b)$ and use the triangle inequality to show $B(b,\rho) \subseteq B(a,r)$. □

Solution. Let $b \in B(a,r)$. We need to find $\rho > 0$ so that $B(b,\rho) \subseteq B(a,r)$. As $b \in B(a,r)$ by the definition of the open ball we have $d(a,b) < r$. Set $\rho = r - d(a,b) > 0$. If $y \in B(b,\rho)$ then $d(b,y) < \rho$ and have by the triangle inequality

$$d(a,y) \leq d(a,b) + d(b,y) < d(a,b) + \rho = d(a,b) + r - d(a,b) - r.$$

Thus $y \in B(a,r)$. As y was an arbitrary element of $B(b,\rho)$ this implies $B(b,\rho) \subseteq B(a,r)$ and we are done. □

Proposition 2. *For any $a \in E$ and $r > 0$ the set $U = \{x \in E : x \notin \overline{B}(a,r)\} = \{x \in E : d(x,a) > r\}$ (that is the compliment of the closed ball $\overline{B}(a,r)$) is open.*

2. Prove this. *Hint:* Let $b \in U$. Then $d(a,b) > 0$. So set $\rho = d(a,b) - r$ and show $B(b,\rho) \subseteq U$. □

Solution. Let $b \in U$, then $d(a,b) > r$ and thus $\rho := d(a,b) - r > 0$. Let $y \in B(b,\rho)$. Then $d(y,b) < \rho = d(a,b) - r$ and thus

$$\begin{aligned} d(a,b) &\leq d(a,y) + d(y,b) \\ &\leq d(a,b) - r + d(y,b) \end{aligned}$$

which implies $d(y,b) > r$. Thus $y \in B(b,\rho)$ implies $y \in U$ and therefore $B(b,\rho) \subseteq U$. So U contains a ball about any of its points and therefore is open. □

Proposition 3. *Let $\{U_\alpha : \alpha \in A\}$ be a possibly infinite collection of open subsets of E. Then the union*

$$U := \bigcup_{\alpha \in A} U_\alpha$$

is open.

This is often stated as the collection of open sets is closed under arbitrary unions.

3. Prove this. *Hint:* Let $a \in U$, then by the definition of union $a \in U_\alpha$ for some $\alpha \in A$. Now use the definition of U_α being open and that $U_\alpha \subseteq U$. □

Solution. Let $a \in U$, as per the hint, this implies, by the definitioin of the union, that $a \in U_\alpha$ for some $\alpha \in A$. As U_α is open there is $r > 0$ so that $B(a,r) \subseteq U_\alpha$. But (definition of union again) $U_\alpha \subseteq U$

and therefore $B(a, r) \subseteq U_\alpha \subseteq U$. So U contains a ball about any of its points and therefore is open. $\square$

Proposition 4. *Let $U_1, U_2, \ldots, U_n \subseteq E$ be a finite collection of open subsets of E. Then the intersection*

$$U = U_1 \cap U_2 \cap \cdots \cap U_n$$

is open.

This is often stated as the collection of open subset is closed under finite intersections.

4. Prove this. *Hint:* Let $a \in U$. Then by definition of the intersection $a \in U_j$ for each j and by definition of U_j being open there is a $r_j > 0$ with $B(a, r_j \subseteq U_j$. Now explain how to chose r so that $B(a, r) \subseteq U$. $\square$

Solution. Let $a \in U$. Then $a \in U_j$ for all $j = 1, 2, \ldots, n$. As U_j is open, there is a $r_j > 0$ with $B(a, r_j) \subseteq U_j$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$. Then for each j we have $B(a, r) \subseteq B(a, r_j) \subseteq U_j$. Since this holds for all j it follows $B(a, r) \subseteq U_1 \cap U_2 \cap \cdots \cap U_n$. Thus the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ contains a ball about any of its points and therefore is open. $\square$

In the next problem you can use the fact that a one element set $\{a\}$ of $\mathbb{R}$ is not open.

5. This problem shows that the collection of open subsets of $\mathbb{R}$ is *not* closed under infinite intersections. Let $U_n = (-1/n, 1/n)$ in $\mathbb{R}$. This is open (as it is the open ball $B(, 1/n)$). Show

$$\bigcap_{n=1}^{\infty} U_n = \{0\}$$

and therefore the intersection is not open. *Hint:* In showing the intersection is just $\{0\}$ you may want to quote a axiom of a well known Greek who died in the battle of Syracuse during Second Punic War. $\square$

Solution. Let $x \in \bigcap_{n=1}^{\infty} U_n$, and assume, toward a contradiction, that $x \neq 0$. Then $|x| > 0$. By the small form of Archimedes' Axiom there is a $n \in \mathbb{N}$ with $1/n < |x|$. This implies $x \notin U_n = (-1/n, 1/n)$ which in turn implies $x \notin \bigcap_{n=1}^{\infty} U_n$. A contrition.