Solutions of some of the problems on homework 7.

Problem 1. Let (E, d) be a metric space and $A \subseteq E$. Let $\overline{A}$ be the set of all points $p \in E$ so that for all $r > 0$ we have $B(p, r) \cap A \neq \varnothing$. Show that $\overline{A}$ is closed. *Hint:* Show the compliment of $\overline{A}$ is open. If q is in the compliment, the writing what this means should come close to finishing the proof. $\square$

Solution. This is a bit tricker than my hint makes it sound. Let $q \in \mathcal{C}(\overline{A})$. Then, as per the hint, by the definition of $\overline{A}$ there is a $r > 0$ with $B(q, r) \cap A = \varnothing$. This is not enough to show $\mathcal{C}(\overline{A})$ is open, for that we need:

Claim. $B(q, r) \cap \overline{A} = \varnothing$. Towards a contradiction assume that this is not the case, then there is a point $x \in B(q, r) \cap \overline{A}$. Then, as $B(q, r)$ is an open set, there is $\rho > 0$ with $B(x, \rho) \subseteq B(q, r)$. This implies

$$B(x, \rho) \cap A \subseteq B(q, r) \cap A = \varnothing.$$

Thus $B(x, \rho) \cap A$. This contradicts that $x \in \overline{A}$.

This for any $q \in \mathcal{C}(\overline{A})$ we have shown there an $r > 0$ with $B(q, r) \subseteq \mathcal{C}(\overline{A})$. Therefore $\mathcal{C}(\overline{A})$ is open and to $\overline{A}$ is closed. $\square$

Problem 2. Let (E, d) be a metric space. Let $S \subseteq E$ with the property that if $s_1, s_2 \in S$ with $s_1 \neq s_2$, then $d(s_1, s_2) \geq 1$. Show S is closed. *Hint:* First show that any ball $B(p, 1/2)$ can contain at most one point of S (use the triangle inequality to show that if $a, b \in B(p, 1/2)$, the $d(a, b) < 1$ and explain why this implies $B(p, 1/2)$ can contain at most one point of S). Let $U = E \setminus S$ be the compliment of S in E and let $p \in U$, that is $p \notin S$. We need to find an $r > 0$ so that $B(p, r) \cap S = \varnothing$.

Case 1. $B(p, 1/2) \cap S = \varnothing$. Then $r = 1/2$ works.

Case 2. $B(p, 1/2) \cap S \neq \varnothing$. Then by what we have just shown, $B(p, 1/2)$ contains exactly one point of S, call it s. Let $r = d(p, s)$ and explain why $B(p, r) \cap S = \varnothing$. $\square$

Solution. If $p \in E$ and $x, y \in B(p, r)$, then $d(x, p) < r$ and $d(y, r) < r$ and so

$$d(x, y) \leq d(x, p) + d(p, y) < r + r = 2r.$$

Letting $r = 1/2$ we have that any two points of a ball $B(p, 1/2)$ are at distance less than 1 from each other. As any two points of S have a distance ≥ 1 from each other any open ball of radius r contains at most one point of S.

Let $U = \mathcal{C}(S)$ be the compliment of S and let $p \in U$. We need to find an open ball about p contained in U.

Case 1. If $B(p, 1/2) \cap S = \varnothing$, then $B(p, 1/2)$ is the required ball.

Case 2. If $B(p, 1/2) \cap S \neq \varnothing$, then let $s \in B(p, 1/2) \cap S$. We have just seen that any open ball of radius $1/2$ contains at most one point of S, so s is the only point of S in $B(p, 1/2)$. Let $r = d(p, s)$. Then $s \notin B(p, r)$. And $r = d(p, s) < 1/2$ as $s \in B(p, 1/2)$ and so $B(p, r) \subseteq B(p, 1/2)$ and $B(p, 1/2)$

contains not point of S other than s. Thus $B(p,r)$ contains no point of s and therefore $B(p,r) \subseteq U$.

So in all cases we have an open ball about p contained in U and so U open and therefore S is closed. $\square$

Problem 3. In the plane $\mathbb{R}^2$, show the half plane $H = \{(x,y) : y > 0\}$ is open. $\square$

Solution. In this problem several of you had trouble with notation. Let $(x_0, y_0) \in H$. By the definition of U we have $y_0 > 0$. Let $r = y_0$. I claim the ball $B((x_0,y_0), r) \subseteq H$. Let $(x,y) \in B((x_0,y_0), r)$. Then we need to show $y > 0$. This is obvious form the picture:

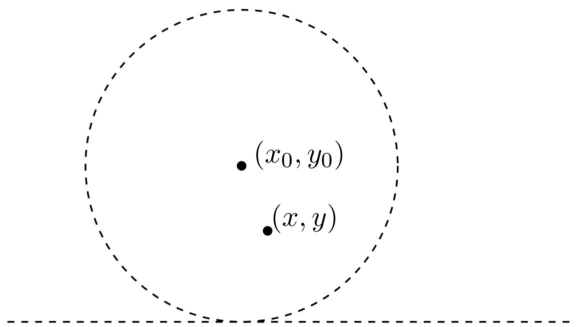

FIGURE 1. The radius of the circle is $r = y_0 > 0$ and if (x,y) is in this circle in it can not have y negative.

But this is analysis so we have to show it using inequalities. This can be done in lots of ways, here is one

$$\begin{aligned}
|y - y_0| &= \sqrt{(y-y_0)^2} \\
&\le \sqrt{(x-x_0)^2 + (y-y_0)^2} \\
&= d((x,y),(x_0,y_0)) && \text{(Def. of distance in } \mathbb{R}^2\text{)} \\
&< r \\
&= y_0
\end{aligned}$$

We now do one of our standard adding and subtracting tricks

$$y = y_0 + (y - y_0) \ge y_0 - |y - y_0| > y_0 - y_0 = 0.$$

Thus y is positive, which shows $(x,y) \in H$. As (x,y) was any point of H this gives $B((x_0,y_0), y_0) \subseteq H$. As (x_0,y_0) was any point of H this shows H is open. $\square$

Problem 4. Let (E,d) be a metric space and $p, p \in E$ with $p \ne q$. Show that $U := \{x \in E : d(p,x) < d(q,x)$ is open. $\square$

Solution. Let $y \in E$. By the reverse triangle inequality the two inequalities

$$|d(q,x) - d(q,y)| < d(x,y)$$
$$|d(p,x) - d(p,y)| < d(x,y)$$

Let $y \in U$. Then $d(q,y) - d(p,y) > 0$. Let

$$r := \frac{d(q,y) - d(p,y)}{2} > 0.$$

If $x \in B(y,r)$, then $d(x,y) < r$ and so

$$\begin{aligned} d(q,x) &= d(q,y) + (d(q,x) - d(q,y)) \\ &\geq d(q,y) - |d(q,x) - d(q,y)| \\ &> d(q,y) - r. \end{aligned}$$

Likewise

$$\begin{aligned} -d(p,x) &= -d(p,y) - (d(p,x) - d(p,y) \\ &\geq -d(p,y) - |d(p,x) - d(p,y)| \\ &> -d(p,y) - r \end{aligned}$$

Therefore

$$\begin{aligned} d(q,x) - d(p,x) &> d(q,y) - r - d(p,r) - r \\ &= d(q,y) - d(p,y) - 2r \\ &= 0. \end{aligned}$$

This implies that at $p \in B(q,r)$ is in U so $B(q,r) \subseteq U$. As q was an arbitrary point of u this implies U is open. $\square$