Mathematics 300 Homework, September 21, 2024.

In light of what we did in class on Friday, let us take a break for just do proofs and do some algebra that will be useful in your later math/science/engineering classes. To start let here is a recap of some of the calculations we did on involving a lot of cancellation. First the well known

$$\begin{aligned}(x-y)(x+y) &= x(x+y) - y(x+y)\\ &= x^2 + xy - xy - y^2\\ &= x^2 - y^2.\end{aligned}$$

Next the somewhat less well known

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x(x^2+xy+y^2) - y(x^2+xy+y^2)\\ &= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3\\ &= x^3 - y^3.\end{aligned}$$

Let us go nuts try a larger example:

$$\begin{aligned}&(x-y)(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5)\\ &\quad = x(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5)\\ &\qquad - y(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5)\\ &\quad = x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5\\ &\qquad\quad - x^5y - x^4y^2 - x^3y^3 - x^2y^4 - xy^5 - y^6 \quad \text{(a massacre happens)}\\ &\quad = x^6 - y^6.\end{aligned}$$

At this point you believe that for any integer that for any integer $k \geq 1$ that

$$\begin{aligned}&(x-y)(x^k + xk - 1y + x^{k-1}y^2 + \cdots + x^2y^{k-2} + xy^{k-1} + y^k)\\ &\qquad = x^{k+1} - y^{k+1}\end{aligned}$$

This is usually stated in (and used) in a "reversed" form. Let $n = k+1$ and then we can factor $x^n - y^n$.

Proposition 1. *Let $n \geq 2$ be an integer. Then for any real numbers we can factor the difference of n-th, $x^n - y^n$, powers as*

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1})$$

Here is how to use this for proving something about congruences (which we also touched on in class).

Proposition 2. *If $a \equiv b \pmod{n}$ then $a^4 \equiv b^4 \pmod{n}$.*

Proof. Assume $a \equiv b \pmod{n}$. Then there is an integer q such that

$$a - b = qn.$$

Now we can use our new knowledge about factoring

$$\begin{aligned} a^4 - b^4 &= (a-b)(a^3 + a^2 b + ab^2 + b^3) \\ &= qn(a^3 + a^2 b + ab^2 + b^3) \qquad\qquad (a - b = qn) \\ &= q'n \end{aligned}$$

where $q' = q(a^3 + a^2 b + ab^2 + b^3) \in \mathbb{Z}$. Thus $a^4 \equiv b^4 \pmod{n}$. □

Hopefully using the factoring formula for $a^k - b^k$ will make the following clear.

Problem 1. Let $k \geq 2$ be an integer and show that $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$. □

Here is a sum related to the above.

Definition 3. A finite ***geometric series*** is a sum of the form

$$S = a + ar + ar^2 + \cdots + ar^n$$

Let use find the sum of these for some small values of n. Start with

$$S = a + ar + ar^2.$$

Multiple by r to get

$$rS = ar + ar^2 + ar^3$$

Then subtract

$$\begin{aligned} S &= a + ar + ar^2 \\ -rS &= \quad - ar - ar^2 - ar^3 \end{aligned}$$

to get

$$(1 - r)S = a - ar^3.$$

Thus if $r \neq 1$ we get

$$S = \frac{a - ar^3}{1 - r}.$$

Let us try the same trick with

$$S = a + ar + ar^2 + ar^3 + ar^4 + ar^5.$$

Multiply by r and subtract

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + ar^4 + ar^5 \\ -rS &= \quad - ar - ar^2 - ar^3 - ar^4 - ar^5 - ar^6 \end{aligned}$$

We have anther massacre and end up with

$$(1 - r)S = a - ar^6$$

so when $r \neq 1$ we can divide to get

$$S = \frac{a - ar^6}{1 - r}.$$

By now you have seen a pattern and so this will not surprise you:

Proposition 4. *Let $r \neq 1$. Then the sum of the finite geometric series*

$$S = a + ar + ar^2 + \cdots + ar^n$$

is

$$S = \frac{a - ar^{n+1}}{1 - r}$$

Problem 2. Be able to prove this on a quiz. □

The number r is the ***ratio***. In the sum

$$S = a + ar + ar^2 + \cdots + ar^n$$

a is the first term and if the series kept going the next term would be ar^{n+1}. I find the easiest way to apply the formula for the sum to the think of it as

$$S = \frac{a - ar^{n+1}}{1 - r} = \frac{\text{first} - \text{next}}{1 - \text{ratio}}.$$

Here are some examples

$$S = 3+3(2)+3(2^2)+3(2^3)+\cdots+3(2^{10}) = \frac{\text{first} - \text{next}}{1 - \text{ratio}} = \frac{3 - 3(2^{11})}{1 - 2} = 6141$$

Here is a slightly more complicated one

$$x^2 + x^4 + x^6 + \cdots + x^{20} = \frac{\text{first} - \text{next}}{1 - \text{ratio}} = \frac{x^2 - x^{22}}{1 - x^2}$$

holds when $x \neq \pm 1$.

Let

$$S = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}.$$

Then

$$\begin{aligned} S &= \frac{1 - \text{next}}{1 - \text{ratio}} \\ &= \frac{1 - (-1/128)}{1 - (-1/2)} = \frac{128 + 1}{128 + 64} = \frac{129}{192}. \end{aligned}$$

For the classical problem[1] of putting one grain rice on the first square of a chess broad, two on the second square, four on the third square, eight on

[1] From the Wikipedia article on putting on grains of rice (or wheat) *Wheat and chessboard problem* `https://en.wikipedia.org/wiki/Wheat_and_chessboard_problem`. The problem appears in different stories about the invention of chess. One of them includes the geometric progression problem. The story is first known to have been recorded in 1256 by Ibn Khallikan.[1] Another version has the inventor of chess (in some tellings Sessa, an ancient Indian Minister) request his ruler give him wheat according to the wheat and chessboard problem. The ruler laughs it off as a meager prize for a brilliant invention, only to have court treasurers report the unexpectedly huge number of wheat grains would outstrip the ruler's resources. Versions differ as to whether the inventor becomes a high-ranking advisor or is executed.

the fourth square: that is doubling the number on each square up until the 64th square, then the total number of grains is

$$1 + 2 + 4 + \cdots + 2^{63} = \frac{1 - 2^{64}}{1 - 2} = 2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615.$$

Remark 5. The internet tells me that "A single long grain of rice weighs an average of 0.001 ounces (29 mg)." Thus the total weight of the rice on the chess board is $(2^{64}-1)/(1{,}000)$ onces. The number of onces $(2^{64}-1)/(1{,}000)$ in a ton is $2{,}000 \times 16 = 32{,}000$. Therefore the weight in tons of the rice

$$W = (2^{64} - 1)/(1{,}000 \times 32{,}000) = 5.76460752303423 \times 10^{11}.$$

The internet also says that the current rate of world rice production is about $P = 7.385477 \times 10^8$ tones/year. At this rate is would take about

$$\frac{W}{P} \approx 780.533$$

years to cover the chess board. □