Mathematics 554 Homework.

You will have a quiz on Monday where have to know the definitions of the following:

- The ***union***, $\bigcup \mathcal{U}$, and ***intersection***, $\bigcup \mathcal{U}$, of a collection, $\mathcal{U}$ of a set E.
- $\mathcal{U}$ is a ***open cover*** of A.
- A is a ***compact*** compact subset of a metric space (E, d).

See pages 72–73 and 74–75 for these definitions.

Here is the definition for compact.

Definition 1. Let A be a subset of the metric space E. Then A is compact if and only if every open cover of A has a finite subcover. □

More explicitly this means that if $\mathcal{U}$ is an open cover of A, then then there is a finite set $\mathcal{U}_0 = \{U_1, U_2, \dots, U_m\}$ of $\mathcal{U}$ such that

$$A \subseteq U_1 \cup U_2 \cup \cdots \cup U_n.$$

Problem 1. Let E be a metric space and $p \in E$. Let $r_1, r_2, \dots, r_m > 0$. Then show

$$B(p, r_1) \cup B(p, r_2) \cup \cdots \cup B(p, r_m) = B(p, r_{\max})$$
$$B(p, r_1) \cap B(p, r_2) \cap \cdots \cap B(p, r_m) = B(p, r_{\min})$$

where

$$r_{\min} = \min(r_1, r_2, \dots, r_m) \quad \text{and} \quad r_{\max} = \max(r_1, r_2, \dots, r_m).$$

Hint: We have done this before, this is just to refresh our memories as we will be using these facts. □

Proposition 2. *Let E be a metric space and $p \in E$. Let $A \subseteq E$ be compact. Then A is bounded, that is there is $r > 0$ such that $A \subseteq B(p, r)$.*

Problem 2. Prove this. *Hint:* Let

$$\mathcal{U} = \{B(p, r) : r > 0\}.$$

Show this is an open cover (again something we have done before). Explain why there is a finite set $\mathcal{U}_0 = \{B(p, r_1), B(p, r_2), \dots, B(p, r_m)\}$ that covers A. Then, by the definition of a cover,

$$A \subseteq B(p, r_1) \cup B(p, r_2) \cup \cdots \cup B(p, r_m).$$

Now use Problem 1. □

Proposition 3. *Let A be a compact subset of a metric space E. Then A is closed in E.*

Problem 3. Prove this. *Hint:* Assume A is compact. To show A is closed it is enough to show that A contains all its adherent points. Towards a contradiction assume A has an adherent point a with $a \notin A$. For each $r > 0$, let

$$U_r = \{x \in E : d(a,x) > r\} = E \setminus \overline{B}(a,r).$$

This is open (we have done this before and you can assume it here). Use that $a \notin A$ to show

$$\mathcal{U} = \{U_r : r > 0\}$$

is a open over of A. (If $p \in A$ then $p \neq a$ so $d(a,p) > 0$. Choose $r < d(a,p)$ and then $p \in U_r$)). Then use compactness to get

$$\mathcal{U}_0 = \{U_{r_1}, U_{r_2}, \dots, U_{r_m}\}$$

that covers A. Show

$$A \subseteq U_{r_1} \cup U_{r_2} \cup \cdots \cup U_{r_m} = U_{r_{\min}}$$

and that this contradicts that a is an adherent point of A. □

We have proven:

Theorem 4. *A subset $A \subseteq \mathbb{R}^n$ of $\mathbb{R}^n$ is sequentially compact if and only if if closed and bounded in $\mathbb{R}^n$.* □

Problem 4. Show that a compact subset of $\mathbb{R}^n$ is sequentially compact. *Hint:* In light of the theorem just quoted, it is enough to show A is closed and bounded. □

The converse of this is true, but we are, at least for the present, going to skip the proof. But the following is true and important.

Theorem 5. *Let A be a subset of $\mathbb{R}^n$. Then the following are equivalent:*

(a) A is closed and bounded.
(b) A is compact.
(c) A is sequentially compact. □

Recall

Definition 6. A function $f \colon E \to E'$ between metric spaces is continuous at $p \in E$ if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d(x,p) < \delta \quad \text{implies} \quad d'(f(x), f(p)) < \varepsilon.$$

We also have the definition

Definition 7. If $f \colon E \to E'$ is a map between metric space, then for $p \in E$ and $q \in E'$

$$\lim_{x \to p} f(x) = q$$

if and only if for all $\varepsilon > 0$, there is a $\delta > 0$ so that

$$0 < d(x,p) < \delta \quad \text{implies} \quad d'(f(x), q) < \varepsilon.$$

Theorem 8. *Let $f \colon E \to E'$ be a map between metric spaces and $p \in E$. Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.*

Problem 5. Prove this. *Hint:* It is mostly just a definition chase. Do not make it hard. □