Mathematics 554 Homework.

We now review a bit of set theory. Let $f\colon E \to E'$ be a map between sets. Recall that if $A \subseteq E$, then the ***image*** of A under f is

$$f[S] = \{f(x) : x \in A\}.$$

And if $B \subseteq E'$ the ***preimage*** of B under f is

$$f^{-1}[B] = \{x \in E : f(x) \in B\}.$$

We recall that taking preimages behaves well with respect to taking unions and intersections.

Proposition 1. *Let $f\colon E \to E'$ be a map between sets and let $\{S_\alpha\}_{\alpha\in I}$ be a collections of subsets of E'. (That is for each $\alpha \in A$ the $S_\alpha \subseteq E'$.) Then*

$$f^{-1}\Big[\bigcup_{\alpha\in I} S_\alpha\Big] = \bigcup_{\alpha\in I} f^{-1}[S_\alpha] \qquad \textit{and}$$
$$f^{-1}\Big[\bigcap_{\alpha\in I} S_\alpha\Big] = \bigcap_{\alpha\in I} f^{-1}[S_\alpha],$$

Proof. To prove the first equality:

$$\begin{aligned} x \in f^{-1}\Big[\bigcup_{\alpha\in I} S_\alpha\Big] &\iff f(x) \in \bigcup_{\alpha\in I} S_\alpha \\ &\iff f(x) \in S_\alpha \quad \text{for at least one } \alpha \in I \\ &\iff x \in f^{-1}[S_\alpha] \quad \text{for at least one } \alpha \in I \\ &\iff x \in \bigcup_{\alpha\in I} f^{-1}[S_\alpha]. \end{aligned}$$

This shows that $f^{-1}\Big[\bigcup_{\alpha\in I} S_\alpha\Big]$ and $\bigcup_{\alpha\in I} f^{-1}[S_\alpha]$ have the same elements and therefore are equal.

Likewise

$$\begin{aligned} x \in f^{-1}\Big[\bigcap_{\alpha\in I} S_\alpha\Big] &\iff f(x) \in \bigcap_{\alpha\in I} S_\alpha \\ &\iff f(x) \in S_\alpha \quad \text{for all } \alpha \in I \\ &\iff x \in f^{-1}[S_\alpha] \quad \text{for all } \alpha \in I \\ &\iff x \in \bigcap_{\alpha\in I} f^{-1}[S_\alpha]. \end{aligned}$$

and therefore $f^{-1}\Big[\bigcap_{\alpha\in I} S_\alpha\Big]$ and $\in \bigcap_{\alpha\in I} f^{-1}[S_\alpha]$ have the same elements and therefore are equal. □

We recall that if S is a subset of some set E then the ***compliment*** of S in E is

$$\mathcal{C}(S) = \{x \in E : x \notin S\}.$$

That is $\mathcal{C}(S)$ is the set of points of E that are not in S. Taking compliments is also well behaved with respect to taking preimages.

Proposition 2. *Let $f\colon E \to E'$ be a map between sets and let $S \subseteq E'$. Then*

$$f^{-1}(\mathcal{C}(S)) = \mathcal{C}(f^{-1}(S)).$$

(Here $\mathcal{C}(S)$ is the compliment of S in E' and $\mathcal{C}(f^{-1}(S))$ is the compliment of $f^{-1}(S)$ in E.)

Problem 1. Prove this. □

We summarize this as

Taking preimages preserves unions, intersections, and compliments.

We now relate continuity of functions to taking preimages of open sets and closed sets.

Lemma 3. *Let $f\colon E \to E'$ be a map between metric spaces. Then the following are equivalent:*

(a) For every open subset $S \subseteq E'$ the preimage $f^{-1}(S)$ is open in E. (That is the preimages of open sets are open.)
(b) For every closed subset $S \subseteq E'$ the preimage $f^{-1}(S)$ is closed in E. (That is the preimages of closed sets are closed.)

Problem 2. Prove this. *Hint:* Let $S \subseteq E'$. Then we have seen that $f^{-1}(\mathcal{C}(S)) = \mathcal{C}(f^{-1}(S))$. Assume that (a) holds, that is that the preimages under f of open sets are open. Let S be closed. Then $\mathcal{C}(S)$ is open and therefore $f^{-1}(\mathcal{C}(S))$ is open. But then $\mathcal{C}(f^{-1}(\mathcal{C}(S))) = f^{-1}(\mathcal{C}(\mathcal{C}(S)))$ is closed. But what is $\mathcal{C}(\mathcal{C}(S))$? This shows that (b) holds and thus that (a) implies (b). Do a similar argument to show that (b) implies (a). □

Theorem 4. *Let $f\colon E \to E'$ be a map between metric spaces. Then then the following are equivalent*

(a) f is continuous.
(b) For every open set $S \subseteq E'$ the preimage $f^{-1}(S)$ is open in E.
(c) For every closed set $S \subseteq E'$ the preimage $f^{-1}(S)$ is closed in E.

Proof. That (b) and (c) are equivalent is the content of Lemma 3. The equivalence of (a) and (b) has been proven in class. □

At first it may not seem that rewriting the condition of f being continuous in terms of preimages of open sets is useful, but we now show that it makes some proofs easy.

Recall that a set in a metric space is connected if and only if it is not the disjoint union of two disjoint nonempty open sets.

Definition 5. Let E be a metric space. Then a ***disconnection*** of E is a pair of open sets U, V in E such that
(a) $E = U \cup V$,
(b) $U \cap V = \varnothing$,
(c) $U, V \neq \varnothing$. □

Definition 6. A metric space is ***connected*** if and only if it is not disconnected. □

Theorem 7. *Let E be a connected metric space and $f\colon E \to E'$ be continuous surjective (that is onto) function. Then E' is connected.*

Problem 3. Prove this. *Hint:* Prove the contrapositive: if E' is disconnected, then E is disconnected. Assume E' is disconnected, say $E' = U \cup V$ with U and V nonempty and disjoint form each other. Then use that f is surjective to show $E = f^{-1}[U] \cup f^{-1}[V]$ and then use the properties of preimages and continuous functions to show this is a disconnection of E. □

We will shortly prove the following:

Theorem 8. *The only connected subsets of $\mathbb{R}$ are the intervals.* □

A basic property of intervals is that if I is an interval and $a, b \in I$ with $a < b$, then I contains all the points between a and b. That is if $a, b \in I$ and $a < y < b$, then $x \in I$.

Theorem 9 (General Intermediate Value Theorem)**.** *Let E be a connected metric space and let $f\colon E \to \mathbb{R}$ be a continuous function. Let $p_0, p_1 \in E$ with $f(p_0) < f(p_1)$. Then for every real number y with $f(p_0) < y < f(p_1)$ there is a $x \in E$ with $f(x) = y$.*

Problem 4. Prove this. *Hint:* Let $E' = f[E]$ be the image of E by f. Then $f\colon E \to E'$ is surjective. By Theorem 7 the set E' is connected. By Theorem 8 this implies E' is connected. The rest should be easy, □

Theorem 10 (Intermediate Value Theorem)**.** *Let $[a, b]$ be a closed interval in $\mathbb{R}$ and $f\colon [a, b] \to \mathbb{R}$ a continuous function with $f(a) \neq f(b)$. Then for every y between $f(a)$ and $f(b)$ the equation $f(x) = y$ has a solution with $a < x < b$.*

Problem 5. (a) Prove this as a corollary of Theorem 9 and the fact that $[a, b]$ is connected by Theorem 8.
(b) Draw some pictures illustrating why the theorem is true. □

The intermediate value theorem is useful in showing that equations have solutions, even in cases where we can not solve them explicitly. Here is an example: the equation $x^7 - 3x + 1 = 0$ has at least some solution with $0 < x < 1$. To see this note that $f(x) = x^7 - 3x + 1$ is continuous on $[0, 1]$. Also $f(0) = 1$ is positive, and $f(1) = -1$ is negative. Therefore by Theorem 10 f takes on the value 0 at some point in $(0, 1)$. That is there there is x_0 with $0 < x_0 < 1$ with $f(x_0) = x_0^7 - 3x_0 + 1 = 0$.

Problem 6. Show that the following have solutions.

(a) $x^3 = \sqrt{7+x}$ on the interval $[0,2]$. *Hint:* This can be rewritten as $x^3 - \sqrt{1+x} = 0$.

(b) $x^3 + 2x + 2 = 0$ on $[-2,2]$.

(c) $x^5 - 4x^3 + x - 9 = 0$ on $[-3,3]$.