Mathematics 300 Homework, October 7, 2024.

The next method of proof we are covering is proof by cases. The most basic example is proving something about all integers by splitting into two case: even and odd.

Example 1. Let n be an integer. Then $n(n+1)$ is even.

Solution. Any integer is either even or odd. There are two cases:

Case 1. n is even. Then $n = 2q$ for some integer q. Thus

$$n(n+1) = (2q)(2q+1) = 2\big(2(2q+1)\big).$$

As $2(2q+1)$ is an integer we have that $n(n+1)$ is twice an integer and thus even.

Case 2. n is odd. Then $n = 2q+1$ for some integer q and

$$n(n+1) = (2q+1)(2q+1+1) = 2\big((2q+1)(q+1)\big)$$

and $(2q+1)(q+1)$ is an integer. So $n(n+1)$ is also twice an integer in this case and therefore even.

□

Here are a couple of problems of this type for you to try.

Problem 1. For any integer n the integer $n^2(n+1)(n+3)$ is divisible by 4. □

Example 2. Let k be an integer. Then $7k^2 - 3k + 5$ is odd.

Solution. Again we use that an integer is either even or odd. But the calculations will be easier if we write these cases as either $k \equiv 0 \pmod 2$ or $k \equiv 1 \pmod 2$. As 7, -3, and 5 are all odd we have

$$7 \equiv 1 \pmod 2, \quad -3 \equiv 1 \pmod 2, \quad 5 \equiv 1 \pmod 2.$$

We now look at the two cases

Case 1. $k \equiv 0 \pmod 2$. Then

$$\begin{aligned} 7k^2 - 3k + 5 &\equiv 1(0)^2 + 1(0) + 1 && \pmod 2 \\ & 1 && \pmod 2 \end{aligned}$$

And $7k^2 - 3k + 5 \equiv 1 \pmod 2$ implies $7k^2 - 3k + 5$ is odd in this case.

Case 2. $k \equiv 1 \pmod 2$. Then

$$\begin{aligned} 7k^2 - 3k + 5 &\equiv 1(1)^2 + 1(1) + 1 && \pmod 2 \\ &\equiv 3 && \pmod 2 \\ &\equiv 1 && \pmod 2 \end{aligned}$$

And again we have $7k^2 - 3k + 5 \equiv 1 \pmod 2$, so that $7k^2 - 3k + 5$ is odd in this case also.

□

Problem 2. Let k be an integer. Let a, b, c be odd integers. Show ak^2+bk+c is odd. □

Problem 3. Let n be an integer. Show $n^4 - 3n^2 + 1$ is odd. *Hint:* The calculations will be much easier if use the cases $n \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$ rather than doing the cases $n = 2q$ and $n = 2q + 1$. □

Anther place where proof by cases comes up is working with absolute values. Recall

$$|x| = \begin{cases} x, & x \geq 0; \\ -x, & x < 0. \end{cases}$$

Problem 4. Read section 3.4 (pages 131–136 in the text). He does a good job both of explaining and giving examples of proof by cases of explaining absolute values and its relation to proof by cases. In the problems on pages 137–140 do problems 10 and 12. □

Problem 5. If you want to play with a few more challenging problems do problems 6, 7, 9. □