Mathematics 300 Homework, October 11, 2024.

Recall that if we do a long division of integers a and b with $b > 0$:

$$\begin{array}{r} q \\ b\overline{)\,a} \\ \ddots \\ \hline r \end{array}$$

where q is the ***quotient*** and r is the remainder, then these are related by

$$a = qb + r \quad \text{with } 0 \leq r < b.$$

This is a basic fact and has been given the name:

The Division Algorithm. *For all integers a and b with $b > 0$, there exist unique integers q and r such that*

$$a = bq + r \text{ and } 0 \leq r < b.$$

$\square$

Question 1 on Monday's quiz will be to state the Division Algorithm.

Here are a couple of problems. Recall that

$$\log_b a = x \quad \text{ if and only if } b^x = a.$$

Proposition 1. $\log_2 3$ *is irrational.*

Proof. Toward a contradiction assume $\log_2 3$ is rational. Then, as $\log_2 3 > 0$ we have that there are positive integers p and q such that

$$\log_2 3 = \frac{p}{q}.$$

This implies

$$2^{\frac{p}{q}} = 3$$

Rise both sides of this to the power q.

$$(2^{\frac{p}{q}})^q = 3^q.$$

That is

$$2^p = 3^q.$$

But this is impossible as 2^p is even and 3^q is odd. So we have the contradiction we needed to complete the proof. $\square$

Problem 1. Let $n \geq 3$ be an odd integer. Prove $\log_2 n$ is irrational. $\square$

Problem 2. Let a, b, c be odd integers. Show that $ax^2 + bx + c = 0$ has no rational roots. *Hint:* Towards a contradiction that there is a rational root,

$$x = \frac{p}{q}$$

in lowest terms so that p and q have no factors in common. This means that at least one of p or q is odd, for if p and q are both even then p and q would have the factor 2 in common. Note if $x = p/q$ is a root, then

$$a\left(\frac{p}{q}\right)^2 + b\frac{p}{q} + b = 0.$$

Multiply by q^2 to get

$$ap^2 + bpq + cq^2 = 0.$$

So it is enough to show this has no solutions with p and q having no common factors. You should be able to finish the proof by considering three cases:

(1) p and q are both odd.
(2) p is odd and q is even.
(3) p is even and q is odd.

(It might simplify calculations to work $\pmod 2$.) $\square$