# Mathematics 300 Homework, October 15, 2024.

**Problem** 1. In the text read section 3.5 pages 141–153. Note that we have already covered much of this material, in particular you are already experts on the results about congruences in this section. In the problem set on pages 153–157 do problems 2, 4(instead of his hint, I would use congruences), 6, and 7.

**Problem** 2. Prove that for any integer $n$ that $n^5 \equiv n \pmod 5$. *Hint:* There are five cases and yes this is a bit messy. □

**Proposition 1.** *Recall that an integer, $n$, is a* ***perfect square*** *if and only if $n = k^2$ for some integer $k$. If $n$ is a perfect square, then*

$$n \equiv 0 \pmod 3 \quad \text{or} \quad n \equiv 1 \pmod 3.$$

*Proof.* Assume $n$ is a perfect square. Then $n = k^2$ for some integer $k$. Modulo 3 there are only three possibilities for $k$:

Case 1. $k \equiv 0 \pmod 3$, then $n = k^2 \equiv 0^2 \equiv 0 \pmod 3$.
Case 2. $k \equiv 1 \pmod 3$, then $n = k^2 \equiv 1^2 \equiv 1 \pmod 3$.
Case 3. $k \equiv 2 \pmod 3$, then $n = k^2 \equiv 2^2 = 4 \equiv 1 \pmod 3$.

This covers all cases and so if $n$ is a perfect square, then $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$ as claimed. □

**Corollary 2.** *If $n \equiv 2 \pmod 3$, then $n$ is not a perfect square.* □

*Proof.* The proposition implies that if $n$ is a perfect square, then $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$, so if $n \equiv 2 \pmod 3$, then $n$ is not a perfect square. □

**Problem** 3. Explain why 1,456,376 is not a perfect square. □

**Problem** 4. Show that if $n$ is a perfect square, then

$$n \equiv 0 \pmod 4 \quad \text{or} \quad n \equiv 1 \pmod 4.$$ □

**Proposition 3.** *Assume the integer $n$ is the sum of two squares. That is $n = a^2 + b^2$ where $a$ and $b$ are integers. Then*

$$n \equiv 0 \pmod 4, \quad n \equiv 1 \pmod 4, \quad \text{or} \quad n \equiv 2 \pmod 4.$$

**Problem** 5. Prove this. *Hint:* The most obvious proof by cases is to let $a$ and $b$ each take on the values 0, 1, 2, and 3. This leads to 16 cases and is more work than is fun. A somewhat cleverer idea is to note if $n = a^2 + b^2$, then by problem 4 each of $a^2$ and $b^2$ can only take on the values 0 and 1. Using this you can reduce to four cases: (1) $a^2 = 0$ and $b^2 = 0$, (2) $a^2 = 1$ and $b^2 = 0$ etc.