Mathematics 300 Homework, October 25, 2024.

Read section 4.1 of the text.

Problem 1. Prove the following using mathematical induction.

(a) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

(b) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

(c) $\dfrac{1}{1\times 2} + \dfrac{1}{2\times 3} + \dfrac{1}{3\times 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

(d) $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$.

Problem 2. Let a be a constant and let

$$y = xe^{ax}.$$

Prove the n derivative of y is

$$\frac{d^n y}{dx^n} = \left(a^n + na^{n-1}\right)e^{ax}.$$

Problem 3. This problem lets you review a bit of calculus.

(a) Let n be a positive integer. Use integration by parts to show

$$\int_0^\infty x^n e^{-x}\,dx = n\int_0^\infty x^{n-1}e^{-x}\,dx.$$

(b) Use Part (a) and induction to so that for all positive integers n the equality

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

holds.

Problem 4. We have seen that the absolute value satisfies the triangle inequality

$$|x+y| \le |x| + |y|$$

for all real numbers x and y. Use induction to show that for any real numbers $x1, x_2, \cdots x_n$ that

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|.$$