Mathematics 300 Homework, August 23, 2024.

Here are a couple of sample propositions and proof as examples of how things should be written. It would also be a good idea to read the section **Writing Guidelines for Mathematics Proofs** starting on Page 22 of the text.

Proposition 1. *If a and b are odd integers, then $a^2 + b$ is even.*

Proof. Because a and b are odd, by devotion of odd there are integers k and ℓ so that

$$a = 2k + 1, \qquad \text{and} b = 2\ell + 1.$$

Then, using some algebra,

$$\begin{aligned} a^2 + b &= (2k+1)^2 + 2\ell + 1 \\ &= 4k^2 + 4k + 1 + 2\ell + 1 \\ &= 4k^2 + 4k + 2\ell + 2 \\ &= 2(2k^2 + 2k + \ell + 1) \\ &= 2q \end{aligned}$$

where $q = 2k^2 + 2k + \ell + 1$ is an integer by closure properties. Thus $a^2 + b$ is even by the definition of even. □

Recall that a ***Pythagorean triple*** is three natural numbers a, b, c with $a^2 + b^2 = c^2$.

Proposition 2. *If a, b, c are a Pythagorean triple, then so are the numbers $A = 3a$, $B = 3b$, $C = 3c$.*

Proof. As a, b, c and 3 are natural numbers so are $A = 3a$, $B = 3b$, and $C = 3c$ by closure properties. Also a, b, and c are a Pythagorean triple and therefore

$$a^2 + b^2 = c^2.$$

Now using some algebra

$$\begin{aligned} A^2 + B^2 &= (3a)^2 + (3b)^2 \\ &= 9a^2 + 9b^2 \\ &= 9(a^2 + b^2) \\ &= 9c^2 && (\text{as } a^2 + b^2 = c^2) \\ &= (3c)^2 \\ &= C^2. \end{aligned}$$

Therefore A, B, and C satisfy the definition of a Pythagorean triple. □

Here are some problems very much like these two.

Problem 1. Prove: If x is odd and y is even then $x^2 + y^2 + 1$ is even. □

Problem 2. If a, b, c is a Pythagorean triple and m is any natural number, then $A = ma$, $B = mb$, and $C = mc$ is also a Pythagorean triple. □

The next problem gives a method of finding lots of Pythagorean triples.

Problem 3. Let p and q be natural numbers with $p > q$. Then

$$\begin{aligned} a &= p^2 - q^2 \\ b &= 2pq \\ c &= p^2 + q^2 \end{aligned}$$

are a Pythagorean triple. □

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Recall that a ***Pythagorean triple*** is three natural numbers a, b, c with $a^2 + b^2 = c^2$.

Proposition 4. *If a, b, c are a Pythagorean triple, then so are the numbers $A = 3a$, $B = 3b$, $C = 3c$.*

Proof. As a, b, c and 3 are natural numbers so are $A = 3a$, $B = 3b$, and $C = 3c$ by closure properties. Also a, b, and c are a Pythagorean triple and therefore

$$a^2 + b^2 = c^2.$$

Now using some algebra

$$\begin{aligned} A^2 + B^2 &= (3a)^2 + (3b)^2 \\ &= 9a^2 + 9b^2 \\ &= 9(a^2 + b^2) \\ &= 9c^2 && \text{(as } a^2 + b^2 = c^2\text{)} \\ &= (3c)^2 \\ &= C^2. \end{aligned}$$

Therefore A, B, and C satisfy the definition of a Pythagorean triple. □

Here are some problems very much like these two.

Problem 4. Prove: If x is odd and y is even then $x^2 + y^2 + 1$ is even. □

Problem 5. If a, b, c is a Pythagorean triple and m is any natural number, then $A = ma$, $B = mb$, and $C = mc$ is also a Pythagorean triple. □

The next problem gives a method of finding lots of Pythagorean triples.

Problem 6. Let p and q be natural numbers with $p > q$. Then

$$\begin{aligned} a &= p^2 - q^2 \\ b &= 2pq \\ c &= p^2 + q^2 \end{aligned}$$

are a Pythagorean triple. □

Problem 7. Do Problem 9 on Page 29 of the text. □

Problem 8. Do Problem 13 on Page 30 of the text. □