Mathematics 300 Homework, November 5, 2024.

We have just started talking about recursion. For use this will mean defining a sequence $a_0, a_1, a_2, \ldots$ be defining a_{n+1} in terms of $a_0, a_1, \ldots, a_n$. Generally our recursions will be of the form

$$a_{n+1} = f(a_n)$$

or

$$a_{n+1} = f(a_n, a_{n-1}).$$

As an example of a recursively defined function let a_n be defined by

$$a_{n+1} = (n+1)a_n, \qquad a_0 = 1.$$

Then we get

$$\begin{aligned} a_1 &= 1a_0 = 1 \\ a_2 &= 2a_1 = 2 \cdot 1 \\ a_3 &= 3a_2 = 3 \cdot 2 \cdot 1 \\ a_4 &= 4a_3 = 4 \cdot 3 \cdot 2 \cdot 1 \\ a_5 &= 5a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ a_6 &= 6a_5 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

Therefore this is just one way to define the factorial function

$$n! = n(n-1)(n-2)\cdots(2)(1).$$

Proposition 1. *Use induction to show the solution to*

$$a_{n+1} = -2a_n + 3 \qquad a_0 = 5$$

is

$$a_n = 4(-2)^n + 1$$

Here is a two step recursion

$$f_{n+1} = f_n + f_{n-1}, \qquad f_1 = f_2 = 1.$$

You can then compute

$$\begin{aligned} f_3 &= f_2 + f_1 = 1 + 1 = 2 \\ f_4 &= f_3 + f_2 = 2 + 1 = 3 \\ f_5 &= f_4 + f_3 = 3 + 2 = 5 \\ f_6 &= f_5 + f_4 = 5 + 3 = 8 \\ f_7 &= f_6 + f_5 = 8 + 5 = 13 \\ f_8 &= f_7 + f_6 = 13 + 8 = 21 \\ f_9 &= f_8 + f_7 = 21 + 13 = 34 \\ f_{10} &= f_9 + f_8 = 34 + 21 = 55 \end{aligned}$$

These are the famous ***Fibonacci numbers***. You should read pages 203–204 of the text about these. It is also worthwhile to watch the TED talk by Art Benjamin about some properties of the Fibonacci numbers:

`https://www.youtube.com/watch?v=SjSHVDfXHQ4`

Problem 1. Use induction to show prove the following formula for the sum of the first n Fibonacci numbers:

$$f_1 + f_2 + \cdots f_n = f_{n+2} - 1$$

Problem 2. Use induction to show that the solution to

$$a_{n+1} = 5a_n - 6a_{n-1}, \qquad a_0 = 5, \quad a_1 = 12$$

is

$$a_n = 2 \cdot 3^n + 3 \cdot 2^n.$$

Here is a useful fact about linear two step recursions.

Problem 3. Let α and β be the solutions to the quadratic equation

$$r^2 = Ar + B$$

where A and B are constants. Show that for any constants C_1 and C_2 that

$$x_n = C_1\alpha^n + C_2\beta^n$$

is a solution to

$$x_{n+1} = Ax_n + Bx_{n-1}.$$

Hint: This proof is just a "plug and chug" problem, no induction needed.

Problem 4. Define a_n by

$$a_{n+2} = \frac{1}{2}\left(a_{n+1} + \frac{2}{a_n}\right) \qquad a_1 = 1, \quad a_2 = 1.$$

(a) Compute a_3, a_4, a_5.
(b) Use induction to prove $1 \leq a_n \leq 2$ for all $n \geq 1$.