Mathematics 300 Homework, November 9, 2024.

Recall the trigonometric addition formulas for sin and cos:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

Let $i = \sqrt{-1}$, so that

$$i^2 = -1.$$

Then complex numbers are of the form $a+bi$. These, as I assume you know, are added by

$$(a+bi) + (c+di) = (a+c) + (b+d)i.$$

and multiplied by the usual rules of algebra:

$$\begin{aligned}(a+bi)(c+di) &= ac + (ad+bc)i + bdi^2\\ &= (ac-bd) + (ad+bc)i\end{aligned}$$

Define $\mathbf{cis}(\alpha)$ by

$$\mathbf{cis}(\alpha) = \cos(\alpha) + i\sin(\alpha).$$

Problem 1. Use the addition formulas for sin and cos to prove

$$\mathbf{cis}(\alpha+\beta) = \mathbf{cis}(\alpha)\,\mathbf{cis}(\beta).$$

□

Problem 2. Use problem 1 and induction to prove

$$\mathbf{cis}(n\alpha) = \mathbf{cis}(\alpha)^n.$$

This is ***de Moivre's formula***. □

Problem 3. Use the formula for cubing binomials:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

to derive formulas for $\cos(3\alpha$ and $\sin(3\alpha)$.

Problem 4. (Optional but fun)

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}$$

(a) Show

$$\alpha^2 = \alpha + 1, \qquad \beta^2 = \beta + 1$$

(That is α and β are roots of the equation $r^2 = r + 1$.)

(b) Define

$$a_n = \frac{\alpha^n + \beta^n}{\sqrt{5}}.$$

Show

$$a_0 = 0, \qquad a_1 = 1$$

and

$$a_{n+1} = a_n + a_{n-1}$$

for $n \geq 1$.

(c) Now use induction to show

$$a_n = f_n$$

where f_n is the n-th Fibonacci number. $\square$

The number α is the ***golden ratio*** and

$$f_n = \frac{\alpha^n + \beta^n}{\sqrt{5}}$$

is called ***Binet's formula***, but was known earlier to Abraham de Moivre and Daniel Bernoulli. Note

$$\alpha \approx 1.61803398875\ldots \qquad \beta \approx -0.61803398875\ldots$$

Thus for $n \geq 1$ we have $|\beta^n/\sqrt{5}| < 1$ and therefore

$$f_n = \text{integer closest to } \frac{\alpha^n}{\sqrt{5}}.$$

which can be used to show the 100-th Fibonacci is

$$f_{100} = 354{,}224{,}848{,}179{,}261{,}915{,}075$$

and the 1,00-th is

$$f_{1{,}000} = 4.346655768\ldots \times 10^{208}.$$