Mathematics 300 Quiz 10 Name: Key

Show your work to get credit.

The problems are worth 10 points each.

1. Define the following:

(a) The integer n is ***even***. There is a $q \in \mathbb{Z}$ such that $n = 2q$

(b) The integer n is ***odd***. There is a $q \in \mathbb{Z}$ such that $n = 2q+1$

(c) The integer a ***divides*** the integer b (in symbols $a \mid b$).

$a, b \in \mathbb{Z}$, $a \neq 0$ and there is $q \in \mathbb{Z}$ such that $b = qa$

(d) For integers a, b, n the a is ***congruent to*** b ***modulo*** n. (in symbols $a \equiv b \pmod{n}$.)

$a, b, n \in \mathbb{Z}$, $n \geq 1$ and there is a q such that $a - b = qn$.

2. Prove or give a counterexample: If n is even, then $(n^3 + 11) \equiv 3 \pmod{8}$.

After checking some examples I guessed it was true.

Proof Assume n is even. Then there is a $k \in \mathbb{Z}$ such that $n = 2k$.

So
$$\begin{aligned}(n^3+11) - 3 &= ((2k)^3 + 11) - 3 \\ &= 8k^3 + 8 \\ &= 8(k^3+1) \\ &= 8q\end{aligned}$$

where $q = k^3 + 1 \in \mathbb{Z}$. Thus we have shown the definition of

$$n^3 + 11 \equiv 3 \pmod{8}$$

holds.

3. (a) Make truth tables for $P \to (P \to Q)$ and $\neg P \wedge Q$.

P	Q	$\neg P$	$\neg P \wedge Q$	$P \to Q$	$P \to (P \to Q)$
T	T	F	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	F	T	T

Not the same

(b) Are $P \to (P \to Q)$ and $\neg P \wedge Q$ logically equivalent? Explain your answer.

No, because they have different truth tables

4. Let P be the statement "If the number a is positive, then the equation $f(x) = a$ has a solution".

(a) What is the converse of this statement?

If the equation $f(x)=a$ has a solution, then the number a is positive.

(b) What is the contrapositive of this statement?

If the equation $f(x)=a$ has no solution, then a is not positive.

(c) What is the negation of this statement?

The number a is positive and the equation $f(x)=a$ has no solution.

5. What is the negation of the statement "Every even number is the sum of two prime numbers"?

Some even number is not the sum of two prime numbers

6. (a) Write $\{x \in \mathbb{Z} : (2x+1)(x-4) = 0\}$ in roster notation. $\{4\}$

(b) $\{x \in \mathbb{Q} : (2x + 1)(x - 4) = 0\}$ in roster notation. $\{-\frac{1}{2}, 4\}$

(c) Write $\{-2, -1, 0, 1, 2, \ldots, 101, 102\}$ in set builder notation. $\{x \in \mathbb{Z} : -2 \leq x \leq 102\}$

7. Prove the transitive law for congruence. That is prove: If $r \equiv s \pmod{m}$ and $s \equiv t \pmod{m}$, then $r \equiv t \pmod{m}$.

Assume $r \equiv s \pmod{m}$ and $s \equiv t \pmod{m}$

Then there are $q_1, q_2 \in \mathbb{Z}$ such that

$$r - s = q_1 m$$
$$s - t = q_2 m.$$

Add these to get

$$r - s + s - t = q_1 m + q_2 m$$
$$r - t = (q_1 + q_2) m = q m$$

where $q = q_1 + q_2 \in \mathbb{Z}$.
Thus we have shown the definition of
$r \equiv t \pmod{m}$ holds

8. Prove: If $a \equiv b \pmod{n}$ and $x \equiv y \pmod{n}$, then $a - x \equiv b - y \pmod{n}$.

Assume $a \equiv b \pmod{n}$ and $x \equiv y \pmod{n}$
Then there are integers q_1, q_2 such that

$$a - b = q_1 n$$
$$x - y = q_2 n.$$

subtract these to get

$$(a - x) - (b - y) = q_1 n - q_2 n = (q_1 - q_2) n = q n$$

where $q = q_1 - q_2 \in \mathbb{Z}$. Thus

$$a - x \equiv b - y \pmod{n}.$$

9. Prove or give a counterexample: If $x \equiv y \pmod{n}$, then $x^2 \equiv y^2 \pmod{n}$.

This is true. Proof Assume

$x \equiv y \pmod{n}$. Then there is a $q \in \mathbb{Z}$

such that

$$x - y = qn.$$

Then

$$x^2 - y^2 = (x-y)(x+y)$$

$$= qn(x-y)$$

$$= q'n$$

where $q' = q(x-y) \in \mathbb{Z}$. Thus

$$x^2 \equiv y^2 \pmod{n}$$

10. Find all Pythagorean triples of the form m, $m+3$, $m+6$.

The triples are: 9, 12, 15.

For these to be a Pythagorian triple we need

$$m^2 + (m+3)^2 = (m+6)^2$$

$$m^2 + \cancel{m^2} + 6m + 9 = \cancel{m^2} + 12m + 36$$

$$m^2 + (6-12)m + (9-36) = 0$$

$$m^2 - 6m - 27 = 0$$

$$(m-9)(m+3) = 0$$

so $m = 9, -3$ are the only solutions. But to be a Pythagorean triple we want $m > 0$, so the only solution that counts is $m = 9$

Thus $m, m+3, m+6 = 9, 12, 15$.