Mathematics 300

Quiz 13 Name: Key

You must show your work to get full credit.

1. What is the remainder when 43,716 is divided by 9?

The remainder is 3

$43{,}716 \equiv 4+3+7+1+6 \pmod 9$

$= 21$

$\equiv 3 \pmod 9$

Theorem. *Let $n \in \mathbb{N}$ be a natural such that $\sqrt{n}$ is not an integer. (That is $n \neq k^2$ for any integer k.) Then $\sqrt{n}$ is irrational.*

We give a proof of this in several steps. To start assume, towards a contradiction, that $\sqrt{n}$ is rational. Then

$$\sqrt{n} = \frac{p}{q}$$

where p and q are positive integers. Then

$$q\sqrt{n} = p.$$

Thus $q\sqrt{n} \in \mathbb{N}$, that is $q\sqrt{n}$ is an integer. Of all the ways to write $\sqrt{n} = p/q$ as a rational number we choose the one where q is smallest. That is

$$q = \text{smallest natural number } k \text{ such that } k\sqrt{n} \text{ is an integer.}$$

The contradiction what ends this proof will be finding a natural number $q^* < q$ such that $q^*\sqrt{n} \in \mathbb{N}$.

As $\sqrt{n}$ is not a whole number (we are assuming $\sqrt{n}$ is not an integer) we can write

$$\sqrt{n} = k + f$$

where $k = \lfloor \sqrt{n} \rfloor$ is the greatest integer in $\sqrt{n}$ and f satisfies

$$0 < f < 1.$$

(That is f is a fraction in the sense it is between 0 and 1.)

2. Show $fq \in \mathbb{N}$. *Hint:* $fq = (\sqrt{n} - k)q$.

$fq = q\sqrt{n} - kq$

We are assuming $q\sqrt{n} \in \mathbb{N}$.

$kq \in \mathbb{N}$ by closure theorems

so

$q\sqrt{n} - kq \in \mathbb{Z}$ as it is the difference of integers

Also $fq > 0$ so

$fq \in \mathbb{N}$

3. Let q^* be the integer $q^* = fq$. Show $q^* < q$.

$q^* = fq < q$ as $f < 1$

4. Show $q^*\sqrt{n} \in \mathbb{N}$ *Hint:* $q^*\sqrt{n} = qf\sqrt{n} = q(\sqrt{n} - k)\sqrt{n}$.

$$
\begin{aligned}
q^*\sqrt{n} &= qf\sqrt{n} \\
&= q(\sqrt{n} - k)\sqrt{n} \qquad (\sqrt{n} = k + f, \text{ so } f = \sqrt{n} - k) \\
&= q\sqrt{n}\sqrt{n} - kq\sqrt{n} \\
&= qn - kq\sqrt{n}
\end{aligned}
$$

we are assuming $q\sqrt{n} \in \mathbb{N}$, so $qn - kq\sqrt{n} \in \mathbb{N}$ by closure unomeres

5. Explain why $q^* < q$ and $q^*\sqrt{n} \in \mathbb{N}$ gives a contradiction.

we choose q to be smallest natual number with $q\sqrt{n} \in \mathbb{N}$,
but we have just shown $q^*\sqrt{n} \in \mathbb{N}$, with $q^* < q$,
contradicting that q is smallest.