Mathematics 300

Quiz 26 **Name:**______________________________

You must show your work to get full credit.

Define a sequence recursively by

$$a_{n+1} = \sqrt{2 + a_n}, \qquad a_0 = \sqrt{2}.$$

Then

$$a_1 = \sqrt{2 + \sqrt{2}}$$
$$a_2 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$
$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$
$$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}$$

1. Prove $a_n < 2$ for all $n \in \mathbb{N}$.

Base case: $n=0$ $\quad a_0 = \sqrt{2} < 2.$

Induction step: Assume $a_n < 2$. Then

$$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2+2} \quad (\text{as } a_n < 2)$$
$$= \sqrt{4}$$
$$= 2.$$

This closes the induction.

2. Prove for $a_{n+1} > a_n$ for $n \in \mathbb{N}$.

Base case $n=0$ $\quad a_{0+1} = \sqrt{2+\sqrt{2}} > \sqrt{2+0} = \sqrt{2} = a_0$

Induction step: Assume $a_{n+1} > a_n$. Then

$$a_{n+2} = \sqrt{2 + a_{n+1}}$$
$$> \sqrt{2 + a_n} \quad (\text{as } a_{n+1} > a_n)$$
$$= a_{n+1}.$$

done