Mathematics 300

Quiz 30 Name: Key

You must show your work to get full credit.

1. Use induction to show the n-derivative of

$$f(x) = \frac{1}{x}$$

is

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}.$$

It will make the calculations easier if we write $f(x) = x^{-1}$

and show $f^{(n)}(x) = (-1)^n n!\, x^{-n-1}$

Base case: $n=1$. $f'(x) = (-1)x^{-2} = (-1)^1 1!\, x^{-1-1}$

so the base case holds

Induction step: Assume $f^{(n)}(x) = (-1)^n n!\, x^{-n-1}$.

take $\frac{d}{dx}$ of both sides to get

$$f^{(n+1)}(x) = \frac{d}{dx}(-1)^n n!\, x^{-(n+1)}$$

$$= (-1)^n n! \,(-(n+1))x^{-(n+1)-1}$$

$$= (-1)^{n+1} n! (n+1)\, x^{-n-2}$$

$$= (-1)^{n+1} (n+1)!\, x^{-(n+1)-1}$$

which closes the induction.

2. (a) Use induction to show: if $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$ for all $n = 1, 2, 3, \ldots$.

Base case: $n=1$ $\quad a^1 \equiv b^1 \pmod{m}$, i.e

$a \equiv b \pmod{m}$.

This is given.

Induction step: Assume $a^n \equiv b^n \pmod{m}$.

Multiple this by the congruence

$a \equiv b \pmod{m}$

to get

$$a \cdot a^n \equiv b \cdot b^n \pmod{m}$$

i.e $\quad a^{n+1} \equiv b^{n+1} \pmod{m}$.

This closes the induction.

(b) Show that $7 | (9^n - 2^n)$ for $n = 1, 2, 3, \ldots$.

We have $9 \equiv 2 \pmod{7}$

so by part (a)

$$9^n \equiv 2^n \pmod{7}$$

which implies

$9^n - 2^n = 7q$ some $q \in \mathbb{Z}$.

so $7 | (9^n - 2^n)$.

3. Show that if McNuggets come in packages of size 4 and 6, then for any it is possible to buy exactly m McNuggets for any even number $m \geq 4$.

For the small cases

$4 = 4$

$6 = 6$

$8 = 4 + 4$

$10 = 4 + 6$

which covers the base case(s). To show true for all even $m \geq 4$, it is enough to show if true for some $m \geq 10$ (as we have done the cases $m \leq 10$), then true for $m+2$.

Induction step Assume we can get a bag of m McNuggets.

Case 1 At least one of our packs has 4 McNuggets. Then take it out and replace with a size 6 pack, to get

$m - 4 + 6 = m + 2.$

Case 2 There are no size 4 packs in our bag of McNuggets. Then all bags are of size 6. Take out a size 6 pack and add in 2 size 4 packs to get

$m - 6 + 4 + 4 = m + 2.$

done

4. Let a_n be defined by the recursion

$$a_{n+1} = \frac{2}{3}a_n + 6, \qquad a_1 = 3.$$

(a) Compute $a_2 =$ 8 $\qquad a_3 =$ $\frac{34}{3}$ $\qquad a_4 =$ $\frac{122}{9}$

$a_1 = \frac{2}{3}a_0 + 6 = \frac{2}{3}(3) + 6 = 2 + 6 = 8$

$a_2 = \frac{2}{3}(8) + 6 = \frac{16}{3} + \frac{18}{3} = \frac{34}{3}$

$a_3 = \frac{2}{3}\left(\frac{34}{3}\right) + 6 = \frac{68}{9} + \frac{54}{9} = \frac{122}{9}$

(b) Prove $a_n < 18$ for all n.

Base case: $a_1 = 3 < 18$, is true.

Induction step. Assume $a_n < 18$.

Then $a_{n+1} = \frac{2}{3}a_n + 6$

$< \frac{2}{3}(18) + 6$ (as $a_n < 18$)

$= 12 + 6 = 18.$

done

(c) Prove $a_{n+1} > a_n$ for all $n \geq 1$.

Base case: $n = 1$. $a_{1+1} = a_2 = 8 > a_1 = 3.$

Induction step Assume $a_{n+1} > a_n$.

Then $a_{n+2} = \frac{2}{3}a_{n+1} + 6$

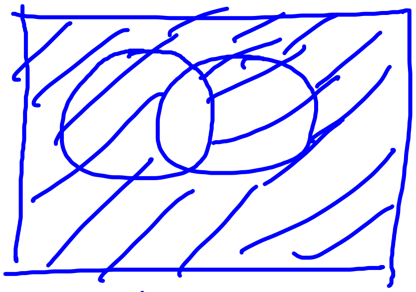
$> \frac{2}{3}a_n + 6$ (as $a_{n+1} > a_n$)

$= a_{n+1}$

done

5. Use Venn diagrams to show each of the following:

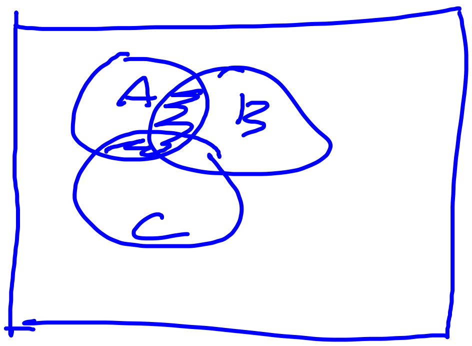
(a) $(A \cap B)^c = A^c \cup B^c$.

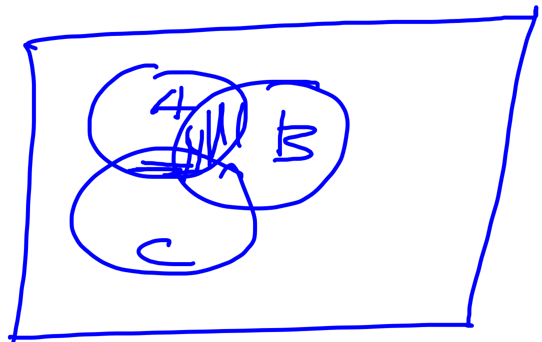


$(A \cap B)^c$

$A^c \cup B^c$

(b) $A \cap (B \cup \cancel{B}\,C) = (A \cap B) \cup (A \cap C)$.

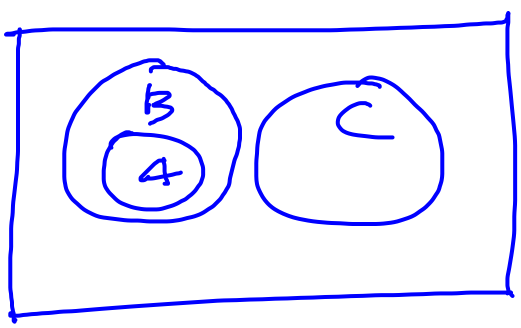




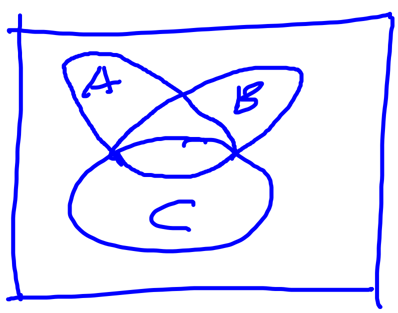
$(A \cap B) \cup (A \cap C)$

6. Draw Venn diagrams showing the following relations between A, B, and C.

(a) $A \subseteq B$ and $B \cap C = \varnothing$.



(b) $A \cap C = B \cap C$.



7. Let

$$A = \{n \in \mathbb{Z} : nis \text{ even }\}$$
$$B = \{x(x+1) : x \in \mathbb{Z}\}$$

(a) Show $B \subseteq A$.
(b) Show A is not a subset of B.

(a) Let $b \in B$. Then $b = x(x+1)$ for some $x \in \mathbb{Z}$.

Case 1 x is even. Then $x = 2q$ for some $q \in \mathbb{Z}$

so $b = 2q(x+1) = 2q'$ where $q' = q(x+1) \in \mathbb{Z}$.

Thus b is even

Case 2 x is odd. Then $x = 2q+1$ for some $q \in \mathbb{Z}$

Then $b = x(2q+1+1) = x(2q+2)$
$= 2x(q+1) = 2q'$

with $q' = x(q+1) \in \mathbb{Z}$. So b is even in this case also.

Thus $b \in B$ implies $b \in A$. So $B \subseteq A$.

(b) $a = 4$ is even. But if $4 \in B$, then $4 = x(x+1)$ some $x \in \mathbb{Z}$.

That is $x^2 + x - 4 = 0$. Solving for x gives

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-4)}}{2} = \frac{-1 \pm \sqrt{17}}{2}$$

which is not an integer. So $4 \notin B$. Thus $A \not\subseteq B$.

8. Let

$$A = \{k \in \mathbb{Z} : k \equiv 1 \pmod 3\}$$
$$B = \{9x + 6y + 4 : x, y \in \mathbb{Z}\}.$$

Prove $A = B$.

Let $b \in B$. Then $b = 9x + 6y + 4$ for some $x, y \in \mathbb{Z}$.

Therefore

$$b = 9x + 6y + 4 \equiv 0x + 0y + 1 \pmod 3$$
$$\equiv 1 \pmod 3$$

so $b \in A$.

Therefore $B \subseteq A$.

Let $a \in A$. Then $a \equiv 1 \pmod 3$

so $a - 1 = 3q$ for some $q \in \mathbb{Z}$.

Then

$$\begin{aligned} a &= 3q + 1 \\ &= 3q - 3 + 4 \\ &= 3(q-1) + 4 \\ &= 9(q-1) - 6(q-1) + 4 \\ &= 9x + 6y + 4 \end{aligned}$$

where $x = q - 1 \in \mathbb{Z}$, $y = -(q-1) \in \mathbb{Z}$.

Therefore $a \in B$. This shows $A \subseteq B$.

So $A \subseteq B$ and $B \subseteq A$. Thus $A = B$.