Mathematics 300

Quiz 31 Name: Key

You must show your work to get full credit.

1. Let $L\colon (0,\infty) \to \mathbb{R}$ satisfy $L(ab) = L(a) + L(b)$. Prove

$$L(a_1 a_2 \cdots a_n) = L(a_1) + L(a_2) + \cdots + L(a_n).$$

We use either $n=1$ or $n=2$ as the base case

$n=1$ $L(a_1) = L(a_1)$ clearly true

$n=2$ $L(a_1 a_2) = L(a_1) + L(a_2)$

which is $L(ab) = L(a) + L(b)$ with $a_1 = a$

$a_2 = b$ so this holds.

Induction step.

Assume $L(a_1 a_2 \cdots a_n) = L(a_1) + L(a_2) + \cdots + L(a_n)$ (I.H.)

Then

$L(a_1 a_2 \cdots a_n a_{n+1}) = L(\underbrace{(a_1 a_2 \cdots a_n)}_{a} \underbrace{a_{n+1}}_{b})$

$= L(a_1 a_2 \cdots a_n) + L(a_{n+1})$ (as $L(ab) = L(a) + L(b)$)

$= L(a_1) + L(a_2) + \cdots + L(a_n) + L(a_{n+1})$ (by I.H.)

This closes the induction

Let A_i with $i \in I$ be a collection of sets. (That is for each $i \in I$ the set A_i is a subset of our universal set U.) Define the union and intersection of these sets by

$$\bigcup_{i \in I} A_i = \{x : \exists i \in I(x \in A_i).\}$$

$$\bigcap_{i \in I} A_i = \{x : \forall i \in I(x \in A_i).\}$$

2. Prove: $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$.

$$\begin{aligned}
\left(\bigcup_{i \in I} A_i\right)^c &= \{x \in U : x \notin \bigcup_{i \in I} A_i\} \\
&= \{x \in U : \neg(\exists i \in I(x \in A_i))\} \\
&= \{x \in U : \forall i \in I(x \notin A_i)\} \\
&= \{x \in U : \forall i \in I(x \in A_i^c)\} \quad \text{(Logic)} \\
&= \bigcap_{i \in I} A_i^c
\end{aligned}$$

3. Prove: $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$.

$$\begin{aligned}
\left(\bigcap_{i \in I} A_i\right)^c &= \{x \in U : x \notin \bigcap_{i \in I} A_i\} \\
&= \{x \in U : \neg(\forall i \in I(x \in A_i)\} \\
&= \{x \in U : \exists i \in I(x \notin A_i)\} \\
&= \{x \in U : \exists i \in I(x \in A_i^c)\} \\
&= \bigcup_{i \in I} A_i^c
\end{aligned}$$