Mathematics 300

Quiz 32 **Name:** Key

You must show your work to get full credit.

Recall that a ***function*** f from a set A to a set B (notation $f\colon A \to B$) is a rule that gives a value $f(x) \in B$ for each $x \in A$. For more details see section 6.1 of the text. Then A is the ***domain*** of the function and B is the ***codomain*** of the function. The ***range*** of f is the set of values of f, that is

$$\operatorname{range}(f) = \{f(x) : x \in A\}.$$

This is also called the ***image*** of f.

Definition. The function $f\colon A \to B$ is ***onto***, also called ***surjective***, if and only if for all $b \in B$ there is an $a \in A$ such that $f(a) = b$. □

One way to think of a function $f\colon A \to B$ as being surjective is that the equation $f(x) = b$ has a solution for all $b \in B$.

Definition. The function $f\colon A \to B$ is ***one to one***, also called ***injective***, if and only if

$$f(a_1) = f(a_2) \quad \text{implies} \quad a_1 = a_2.$$

Being injective is equivalent to

$$a_1 \neq a_2 \quad \text{implies} \quad f(a_1) \neq f(a_2).$$

1. Prove the two definitions of injective are equivalent. *Hint:* Contrapositive.

The contrapositive of
"$f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$"
is "$a_1 \neq a_2 \Longrightarrow f(a_1) \neq f(a_2)$"
so they are equivalent.

2. Show the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^2$$

is neither injective or surjective. What is the range of f? □

It is not injective because
$f(-1) = f(1) = 1$, but $-1 \neq 1$.

It is not surjective because
$f(x) = x^2 = -1$ has no solution.

The range of f is
$$\operatorname{range}(f) = \{x^2 : x \in \mathbb{R}\} = [0, \infty)$$
because any $y \geq 0$ has a square root
so $y = f(\sqrt{y}) = (\sqrt{y})^2$.

3. Show the function $f\colon \mathbb{R} \to [0, \infty)$ is surjective, but not injective. ☐

As before it is not injective, as
$f(-1) = f(1) = 1$.
But it is surjective as
range$(f) = [0,\infty)$ and $[0,\infty)$ is the codomain.

4. Show the function $f\colon [0, \infty) \to [0, \infty)$ is both surjective, and injective ~~injective~~. ☐

It is injective because if $a, b \geq 0$ and $f(a) = f(b)$, that is $a^2 = b^2$, then
$a^2 - b^2 = (a+b)(a-b) = 0$.
If $a = 0$ then $0^2 = 0 = b^2$, so also $b = 0$.
If $a \neq 0$, then $a > 0$ so $a + b > 0$ and
so $(a+b)(a-b) = 0 \Rightarrow a - b = 0 \Rightarrow a = b$.
Thus injective.
As any $y \geq 0$ has $\sqrt{y} \geq 0$ then
$f(\sqrt{y}) = (\sqrt{y})^2 = y$. So any $y \in [0,\infty)$ is in the range of f. Thus f surjective

Definition. A function that is both surjective and injective is ***bijective***. ☐

5. Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = mx + b$$

where m, and b are constants with $m \neq 0$. Prove g is bijective. ☐

<u>Injective</u> If $g(x_1) = g(x_2)$, then
$$m x_1 + \cancel{b} = m x_2 + \cancel{b}$$
$$m x_1 = m x_2$$
$$x_1 = x_2.$$
Thus $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$
<u>surjective</u> Let $y \in \mathbb{R}$. Then
$$g(x) = mx + b = y$$
has the solution
$$x = \frac{y-b}{m}.$$
Thus g is surjective.