Mathematics 300 Test 1 Name: Key

Show your work to get credit.

The problems are worth 10 points each.

1. (10 points) Define the following:

(a) The integer n is ***even***.

There is an integer q such that $n=2q$

(b) The integer n is ***odd***.

There is an integer q such that $n=2q+1$

(c) The integer a ***divides*** the integer b (in symbols $a \mid b$).

$a \neq 0$ and there is an integer q such that $b = qa$.

(d) For integers a, b, n the a is ***congruent to*** b ***modulo*** n. (in symbols $a \equiv b \pmod{n}$.)

$n > 0$ and there is an integer q such that $a - b = qn$

2. (5 points) Prove or give a counterexample: If n is an integer and $9 \mid n^2$, then $9 \mid n$. (If you give a counterexample, your answer should be at least one complete sentence explaining why it is a counterexample.)

Counterexample $n=6$. Then $9 \mid n^2 = 36$, but $9 \nmid n = 6$.

3. (10 points) Prove or give a counterexample: If $n \equiv -1 \pmod 6$, then $n^2 \equiv 1 \pmod 6$. (If you give a counterexample, your answer should be at least one complete sentence explaining why it is a counterexample.)

This is true. Assume $n \equiv -1 \pmod 6$.

Then $n-(-1) = n+1 = 6q$

for some integer q so

$$n = 6q-1$$

and

$$n^2 = (6q-1)^2 = 36q^2 - 12q + 1$$
$$= 6(6q^2 - 2q) + 1$$
$$= 6Q + 1$$

where $Q = 6q^2 - 2q \in \mathbb{Z}$ by closure properties.

so

$$n^2 - 1 = 6Q \quad \text{with } Q \in \mathbb{Z}$$

Thus $n^2 \equiv 1 \pmod 6$

4. (10 points) (a) Make a truth table for $P \to (P \to Q)$.

P	Q	$P \to Q$	$P \to (P \to Q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

same

(b) Is $P \to (P \to Q)$ logically equivalent to $P \to Q$? Explain your answer.

Yes the two have the same truth table

5. (15 points) For the statement:

If a number is green, then it is even.

(a) Give the converse.

If a number is even, then it is green

(b) Give the contrapositive.

If a number is odd, then it is not green

(c) Give the negation.

The number is green, but not even.

6. (5 points) What is the negation of the statement:

All blue dogs vote for red cats

Some blue dog does not vote for red cats.

7. (5 points) A subset $S \subseteq \mathbb{R}$ is ***bounded above*** if and only if there is $b \in \mathbb{R}$ such that $x \leq b$ for all $x \in S$. What does in mean for S to not be bounded above? (That is what is the negation of the statement "There is $b \in \mathbb{R}$ such that $x \leq b$ for all $x \in S$".

For all $b \in \mathbb{R}$ such that $x > b$ for some $x \in S$

8. (10 points) Prove: If a and b are integers with a odd and $b \equiv 2 \pmod 3$, then $(a+5)(b+1)$ is divisible by 6.

Assume a is odd and $b \equiv 2 \pmod 3$. Then there are $m, n \in \mathbb{Z}$ so that

$$a = 2m+1, \quad b-2 = 3n.$$

I.e. $a = 2m+1$, $b = 3n+2$

Then

$$\begin{aligned}(a+5)(b+1) &= (2m+1+5)(3n+2+1) \\ &= (2m+6)(3n+3) \\ &= (2(m+3))(3(n+1)) \\ &= 6(m+3)(n+1) \\ &= 6q\end{aligned}$$

where $q = (m+3)(n+1) \in \mathbb{Z}$.

So 6 divides $(a+5)(b+1)$

done

9. (10 points) Prove: If n^7 is odd, then n is odd.

We prove the converse: If n is even, then n^7 is even.

Assume n is even. Then $n = 2q$ for some $q \in \mathbb{Z}$. So

$$n^7 = (2q)^7 = 2(2^6 q^7) = 2Q$$

where $Q = 2^6 q^7 \in \mathbb{Z}$. Thus n^7 is even.

done

10. (10 points) Prove: If $x \equiv a \,(\mathrm{mod}\ n)$, then $x^2 + x \equiv a^2 + a \,(\mathrm{mod}\ n)$.

Assume $x \equiv a \pmod{n}$. Then there is a $q \in \mathbb{Z}$ with

$$x - a = qn.$$

$$\begin{aligned}(x^2+x) - (a^2+a) &= (x^2-a^2) + (x-a) \\ &= (x-a)(x+a) + (x-a) \\ &= (x-a)(x+a+1) \\ &= qn\,(x+a+1) \qquad (\text{as } x-a = qn) \\ &= nQ\end{aligned}$$

where $Q = q(x+a+1) \in \mathbb{Z}$.

so $x^2 + x \equiv a^2 + a \pmod{n}$

11. (10 points) (a) Write $A = \{x \in \mathbb{N} : x^2 = 16\}$ in roster notation.

$x^2 = 16 \Rightarrow x = \pm 4$. But $-4 \notin \mathbb{N}$.

$A =$ $\{4\}$

(b) Write $B = \{-7, -6, \ldots, 5, 6\}$ is set builder notation.

$B =$ $\{n \in \mathbb{Z} : -7 \leq n \leq 6\}$