Mathematics 300 Test 3 Name:________________________

Show your work to get credit.

1. (10 points) We have shown that the absolute value of real numbers satisfies

$$|x+y| \le |x| + |y|$$

Use this fact and induction to prove

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|. \tag{1}$$

Solution. We are given

$$|x+y| \le |x| + |y| \qquad \text{(Given)}$$

We can use for our base case either $n = 1$, in which case (1) becomes

$$|x_1| \le |x_1|$$

which is true. Or we could us $n = 2$ when (1) becomes

$$|x_1 x_2| \le |x_1| + |x_2|$$

which is just (Given) with $x = x_1$ and $y = x_2$. So this is true.

Induction step. Assume

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n| \qquad \text{(I.H.)}$$

Now we use our grouping trick:

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &= |\overbrace{x_1 + x_2 + \cdots + x_n}^{x} + \overbrace{x_{n+1}}^{y}| \\ &\le |\overbrace{x_1 + x_2 + \cdots + x_n}^{x}| + |\overbrace{x_{n+1}}^{y}| && \text{(by Given)} \\ &\le |x_1| + |x_2| + \cdots |x_n| + |x_{n+1}| && \text{(by (I.H.)).} \end{aligned}$$

which completes the induction. □

2. (10 points) Let $p(x)$ be a polynomial and set $f(x) = xp(x)$.
Prove the n-derivative of $f(x)$ is $f^{(n)}(x) = xp^{(n)} + np^{(n-1)}(x)$.

Solution. Using the product rule we have

$$f'(x) = xp'(x) + x'p(x) = xp^{(1)}(x) + p^{(0)}(x)$$

which is the $n = 1$ case of what we want to prove. This is our base case.

For the induction step assume

$$f^{(n)}(x) = xp^{(n)} + np^{(n-1)}(x) \qquad \text{(I.H.)}$$

which is our induction hypothesis. We now take the derivative of both sides of this and use the product rule

$$\begin{aligned} f^{(n+1)}(x) &= \left(f^{(n)}(x)\right)' \\ &= \left(xp^{(n)}(x) + np^{(n-1)}(x)\right)' \\ &= xp^{(n+1)}(x) + 1p^{(n)}(x) + np^{(n)}(x) \\ &= xp^{(n+1)}(x) + (n+1)p^{(n)}(x) \end{aligned}$$

which is (I.H.) with n replaced by $(n+1)$. So the induction is complete. □

3. (10 points) Chicken nuggets come in boxes with either 3 nuggets or 4 nuggets. Prove that for any $n \ge 8$ that is possible to buy boxes so that the total number of nuggets is n.

Solution. We are given that the base case is $n = 4$. Then buy two boxes of size 4 to get

$$8 = 4 + 4$$

nuggets. Thus the base case holds.

For the induction step assume that we have bought boxes that contain exactly n nuggets with $n \geq 8$. There are two cases (and there are other ways to do the cases).

Case 1. We have at least one box with 3 nuggets. Then we trade in a box of size 3 for a box of size four giving us

$$n - 3 + 4 = n + 1$$

nuggets.

Case 2. We have no boxes of size 3. Then all the boxes are of size 4. As $n \geq 8$ there are at least two boxes of size 4. So we trade in two boxes of size 4 for three boxes of size 3. This gives us

$$n - 2(4) + 3(3) = n + 1$$

nuggets.

This covers all cases and completes the induction. □

4. (15 points) (a) We have shown that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$. Use this fact and induction to prove: If $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$ for all positive integers n.

Solution. The base case is $n = 1$. In this case the proposition

$$a \equiv b \pmod{m} \quad \text{implies} \quad a^1 \equiv b^1 \pmod{m}$$

which is true.

For the induction step assume

$$a \equiv b \pmod{m} \quad \text{implies} \quad a^n \equiv b^n \pmod{m} \qquad \text{(I.H.)}$$

Then let $c = a^n$ and $d = b^n$ in our given proposition "$a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$" to conclude

$$a \equiv b \pmod{n} \quad \text{implies} \quad aa^n \equiv bb^n \pmod{m}$$

that is

$$a \equiv b \pmod{n} \quad \text{implies} \quad a^{n+1} \equiv b^{n+1} \pmod{m}$$

which completes the induction. □

(b) Use part (a) of this problem to show $7 \mid (10^n - 3^n)$ for all $n \in \mathbb{N}$.

Solution. Note

$$10 \equiv 3 \pmod{7}.$$

Therefore by Part (a)

$$10^n \equiv 3 \pmod{7}$$

which implies $7 \pmod{()} 10^n - 7^n)$. □

5. (15 points) Define $f(n)$ recursively by

$$f(n+1) = 2f(n) - 3 \qquad f(0) = 8.$$

(a) Compute the following:

$f(1) =$ ________ $f(2) =$ ________ $f(3) =$ ________

Solution.

$$f(1) = 2f(0) - 3 = 2(8) - 3 = 16 - 3 = 13$$
$$f(2) = 2f(1) - 3 = 2(13) - 3 = 26 - 3 = 23$$
$$f(3) = 2f(2) - 3 = 2(23) - 3 = 46 - 3 = 43.$$

□

(b) Prove $f(n) = 5(2)^n + 3$.

Solution. To start

$$5(2)^0 + 3 = 5(1) + 3 = 8 = f(0).$$

Thus the base case of $n = 0$ holds.

For the induction step assume

$$f(n) = 5(2)^n + 3 \qquad \text{(I.H.)}$$

Then

$$\begin{aligned} f(n+1) &= 2f(n) - 3 \\ &= 2(5(2^n + 3)) - 3 && \text{(by (I.H.))} \\ &= 5(2)^{n+1} + 6 - 3 \\ &= 5(2)^{n+1} + 3 \end{aligned}$$

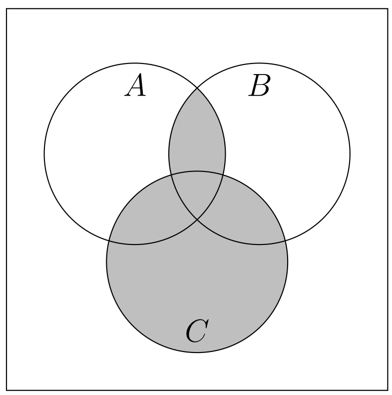
which is (I.H.) with n replaced by $(n+1)$ and we are done. □

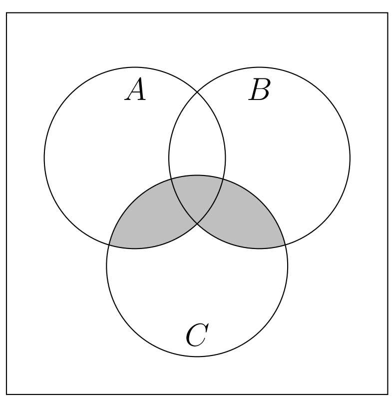
6. (10 points) True or False (circle one): For any sets A, B, and C

$$(A \cap B) \cup C = (A \cup B) \cap C.$$

Use Venn diagrams to explain your answer.

Solution. This is **False**.

The Venn diagram for $(A \cap B) \cup C$ is 

The Venn diagram for $(A \cup B) \cap C$ is 

They are not equal so $(A \cap B) \cup C \neq (A \cup B) \cap C$. □

7. (10 points) Let

$$\begin{aligned} A &= \{1, 2, 3, 4\} \\ B &= \{-2, -1, 0, 1\} \\ C &= \{0, 2\} \end{aligned}$$

What are the following:

$A \cup B =$ ____________________

$(A \cap B) \cup C =$ ____________________

$(A \cap B) - C =$ ____________________

Solution.

$$\begin{aligned} A \cup B &= \{-2, -1, 0, 1, 2, 3, 4\} \\ (A \cap B) \cup C &= \{1, 2, 3\} \\ (A \cap B) - C &= \{1\}. \end{aligned}$$

□

8. (10 points) Let $A = \{n \in \mathbb{Z} : 3 \mid n\}$ and $B = \{x(x+1) : x \in \mathbb{Z}\}$. Show $B \not\subseteq A$ (that is B is not a subset of A).

Solution. To show that $B \not\subseteq A$ we need to find an element $b \in B$ with $b \notin A$. If $x = 1$, then $x(x+1) = 1(1+1) = 2$. So $2 \in B$. But 3 does not divide 2 so $2 \notin A$. Thus the element $b = 2$ shows B is not a subset of A. □

9. (10 points) Let

$$\begin{aligned} A &= \{k \in \mathbb{Z} : 2 \mid k\} \\ B &= \{4x + 6y : x, y \in \mathbb{Z}\} \end{aligned}$$

Prove $A = B$.

Solution. We first show $B \subseteq A$. Let $b \in B$. Then $b = 4x + 6y$ for some integers $x, y \in \mathbb{Z}$. Then $b = 2(2x + 3y)$ and $2x + 3y \in Z$ by closure properties. Therefore $2 \mid b$ which implies $b \in A$. Thus any element of B is an element of A, which implies $B \subseteq A$.

Next we show $A \subseteq B$. Let $a \in A$. Then $2 \mid a$, and therefore $a = 2q$ for some $q \in \mathbb{Z}$. Then

$$a = 2q = 4(-q) + 6(q) = 4x + 6y$$

where $x = -q \in \mathbb{Z}$ and $y = q \in \mathbb{Z}$. Thus shows $a \in B$. As a was any element of A this implies $A \subseteq B$.

Thus we have $A \subseteq B$ and $B \subseteq A$ and therefore $A = B$ as required. □