Mathematics 554 Homework.

There will be a quiz on Monday where you have to know the definitions of the following:

(a) The definitions of a sequence of real numbers $\langle x_n \rangle_{n=1}^{\infty}$ being
   (i) ***increasing***,
   (ii) ***monotone increasing***,
   (iii) ***decreasing***,
   (iv) ***monotone decreasing***, and
   (v) ***bounded***.

(b) The definition of a sequence $\langle p_n \rangle_{n=1}^{\infty}$ in a metric space being a ***Cauchy sequence***.

(c) The sequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$ is a ***subsequence*** of the sequence $\langle p_n \rangle_{n=1}^{\infty}$.

We did the dual version (monotone increasing replaced by monotone decreasing) in class. But the ideas involved important and worth revisiting.

Problem 1. Let $\langle a_n \rangle_{n=1}^{\infty}$ be a monotone sequence of real numbers which is bounded above. Show that the sequence converges. *Hint:* Let $S = \{a_n : n = 1, 2, 3, \ldots\}$. One of our hypothesis is that S is bounded above. Thus, by the least upper bound axiom, this set has a least upper bound. Let $L = \sup(S)$. Let $\varepsilon > 0$. Then $L - \varepsilon < L$ and L is the least upper bound and therefore $L - \varepsilon$ is not an upper bound for S. Explain why this implies there is an N such that $L - \varepsilon < a_N \leq L$. Now use that the sequence is monotone increasing to show $n \geq N$ implies $L - \varepsilon < a_N \leq a_n \leq L$ and that this implies $|L - a_n| < \varepsilon$. □

Theorem 1. *Every sequence $\langle a_n \rangle_{n=1}^{\infty}$ of real numbers has a monotone subsequence.*

Problem 2. Prove this. *Hint:* Here is an outline of what seems to be the easiest way to prove this. Call a_n a ***peak point*** if and only if $a_n \geq a_m$ for all $m \geq n$. That is if a_n is a peak point and $m \geq n$ then $a_m \leq a_n$. Now split the proof into two cases.

Case 1. There are infinitely many peak points. In this case explain why there are $n_1 < n_1 < n_3 < \cdots$ such that each a_{n_k} is a peak point and why this implies the sequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$ is monotone decreasing.

Case 2. There are only finitely many peak points. If a_{n_0} is the peak point a_n with largest subscript n, then if $n > n_0$ the point a_n is not a peak point. Let $n_1 > n_0$. Then a_{n_1} is not a peak point and therefore there is a $n_2 > n_1$ with $a_{n_2} > a_{n_1}$. But a_{n_2} is not a peak point (why?) and therefore there is a $n_3 > n_2$ with $a_{n_3} > a_{n_2}$. Likewise a_{n_3} is not a peak point so there is a $n_4 > n_3$ with $a_{n_4} > a_{n_3}$. Continuing in this manner we get a sequence $\langle a_{n_k} \rangle_{n=1}^{\infty}$ which is increasing. □

Theorem 2. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in a metric space (E, d). Assume that this sequence has a convergent subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$. Then the original sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges.*

Put more succinctly: Cauchy plus convergent subsequence implies convergent.

Problem 3. Prove Theorem 2. *Hint:* The $\langle p_{n_k}\rangle_{k=1}^{\infty}$ converges, so there is a $p \in E$ such that

$$\lim_{k\to\infty} a_{n_k} = p.$$

The basic idea of the proof is to note that by the triangle inequality that

$$d(p, p_n) \le d(p, p_{n_k}) + d(p_{n_k}, p_n).$$

Because the subsequence converges, the first term on the right side of the inequality can be made small by making k large. Because the original sequence is Cauchy the second term on the right can be made small by making both n_k and n large. Here is an outline of making this precise. Let $\varepsilon > 0$.

(a) Explain why there is a N such that

$$m, n \ge N \quad \text{implies} \quad d(a_m, a_n) < \frac{\varepsilon}{2}.$$

(b) Explain there is a $n_k \ge N$

$$d(p, p_{n_k}) < \frac{\varepsilon}{2}.$$

(c) Show

$$n \ge N \quad \text{implies} \quad d(p, p_n) < \varepsilon$$

which completes the proof. □

Proposition 3. *Let $\langle a_n\rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Then $\langle a_n\rangle_{n=1}^{\infty}$ is bounded.*

Problem 4. Prove this. *Hint:* The proof is not much different that the proof showing that a convergent sequence is bounded.

Theorem 4. *Show that every Cauchy sequence in $\mathbb{R}$ converges.*

Problem 5. Prove this. *Hint:* Put the following pieces together by quoting the correct results: A Cauchy sequence is bounded. Every sequence of real numbers has a monotone subsequence. A bounded monotone sequence of real numbers converges. A Cauchy sequence with a convergent subsequence converges. □