Mathematics 554 Homework.

The following is based on Section 3.5, pages 70–72, which you should read.

Theorem 1 (Bolzano-Weierstrass Theorem for Sequences)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Problem 1. Prove this. *Hint:* Put together the following facts to get a proof.

(a) Every sequence in $\mathbb{R}$ has a monotone subsequence.
(b) Bounded monotone sequences in $\mathbb{R}$ converge. □

Definition 2. A subset, S, of a metric space is ***sequentially compact*** if and only if every sequence $\langle p_n\rangle_{n=1}^{\infty}$ of points from S has a subsequence that converges to a point of S. □

Theorem 3 (Bolzano-Weierstrass Theorem)**.** *Every closed bounded subset of a $\mathbb{R}$ is sequentially compact.*

Problem 2. Prove this. *Hint:* You should be able to get a proof of this by putting together the follow results we have already proven.

(a) Every sequence of real numbers has a monotone subsequence.
(b) Bounded monotone sequences are convergent.
(c) If a set is closed, it contains the limits of all its convergent subsequence. □

Lemma 4. *Let $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n)\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^2$. Then $\langle p_n\rangle_{n=1}^{\infty}$ if and only if both the sequences*

$$\langle x_n\rangle_{n=1}^{\infty}, \qquad \langle y_n\rangle_{n=1}^{\infty}$$

converge.

Proof. We have done a version of this in class, and it is also proven on pages 70-71 of the notes. □

Lemma 5. *Let $\langle p_n\rangle_{n=1}^{\infty}$ be a convergent sequence in a metric space. Say $\lim_{n\to\infty} p_n = p$. Let $\langle p_{n_k}\rangle_{k=1}^{\infty}$ be a subsequence. Then also*

$$\lim_{k\to\infty} p_{n_k} = p.$$

Problem 3. Prove this. *Hint:* Let $\varepsilon > 0$. As $\lim_{n\to\infty} p_n = p$ there is a N such that

$$n \geq N \quad \text{implies} \quad d(p_n, p) < \varepsilon.$$

But $n_k \geq k$ and therefore if $k \geq N$ we have $n_k \geq N$. It should now be easy. □

Theorem 6. *Every closed bounded subset of $\mathbb{R}^n$ is sequentially compact.*

Problem 4. Prove this for $n = 2$. *Hint:* Let S be a closed bounded subset of $\mathbb{R}^n$ and let $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n)\rangle_{n=1}^{\infty}$ be a sequence of points from S. Then show the following

(a) The sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ are both bounded in $\mathbb{R}$.
(b) The sequence $\langle x_n \rangle_{n=1}^{\infty}$ has a convergent subsequence $\langle s_{n_k} \rangle_{n=1}^{\infty}$, say $\lim_{k\to\infty} x_{n_k} = a$.
(c) The sequence $\langle y_{n_k} \rangle_{k=1}^{\infty}$ has a convergent subsequence $\langle y_{n_{k_j}} \rangle_{j=1}^{\infty}$.
(d) The subsequence $\langle p_{n_{k_j}} \rangle_{j=1}^{\infty}$, say $\lim_{j\to\infty} p_{n_{k_j}} = b$. of $\langle p_n \rangle_{n=1}^{\infty}$ converges:

$$\lim_{j\to\infty} p_{n_{k_j}} = (a, b).$$

(e) As S is closed, it contains the limits of all its convergent subsequence. Explain why this implies $(a, b) \in S$.
(f) Explain why we are done. □