Mathematics 554 Homework.

1. Continuous functions between metric spaces

Recall our basic result how continuous functions between metric spaces are wonderful.

Theorem 1 (Continuous functions are wonderful)**.** *Let $f\colon E \to E'$ be a map between metric spaces. Then the following are equivalent:*

(a) f is continuous (that its the ε, delta definition of continuity holds at each point $p \in E$.

(b) f does the right thing to limits:

$$lim_{x\to p} f(x) = f(p)$$

for all $p \in E$.

(c) f does the right thing to limits of convergent sequences: if

$$\lim_{n\to\infty} x_n = p$$

in E, then

$$\lim_{n\to\infty} f x_n = f(p)$$

in E'.

(d) The preimage of open sets are open: If $V \subseteq E'$ is open in E' then

$$f^{-1}[V] = \{x \in E : f(x) \in V\}$$

is open E.

(e) The preimage of closed sets is closed: If $A \subseteq V$ is closed, then

$$f^{-1}[A] = \{x \in E : f(x) \in A\}$$

is closed in E'. □

2. Continuous functions on compact sets.

Theorem 2 (Continuous images of compact sets are compact)**.** *Let $f\colon E \to E'$ be continuous and $K \subseteq E$ compact. Then the image*

$$f[K] = \{f(x) : x \in K\}$$

is also compact.

Proof using sequences. We will use that a set is compact if and only if it is sequentially compact. Assume K compact and that f is continuous. Let $\langle y_n\rangle_{n=1}^\infty$ be a sequence in $f[K]$. To show $f[K]$ compact we need to show this sequence has a convergent subsequence. By the definition of the image $f[K]$ each $y_n \in f[K]$ is of the form $y_n = f(x_n)$ for some $x_n \in K$. As K is compact the sequence $\langle x_n\rangle_{n=1}^\infty$ has a subsequence $\langle x_{n_k}\rangle_{k=1}^\infty$ which converges to some point x of K. That is $\lim_{k\to\infty} x_{n_k} = x$ with $x \in K$. But f is continuous and therefore does the right thing to limits. Thus, using that $y_{n_k} = f(x_{n_k})$,

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(x) \in f[K],$$

where $f(x) \in K$ because $x \in K$. Therefore the sequence $\langle y_n \rangle_{n=1}^{\infty}$ has the subsequence $\langle y_{n_k} \rangle_{k=1}^{\infty}$ which converges to the point $f(x) \in f[K]$. Therefore $f[K]$ is compact (as it is sequentially compact). □

Proof using open covers. Let $K \subseteq E$ be compact and let $\mathcal{V}$ be an open cover of the image $f[K]$. We wish to show $\mathcal{V}$ has a finite subcover. Let

$$\mathcal{U} = \{f^{-1}[V] : V \in \mathcal{V}\}.$$

As f is continuous and the preimage of open sets by continuous functions are open, each $f^{-1}[V]$ is open. Also if $x \in K$, then $f(x) \in f[K]$ and $\mathcal{V}$ covers $f[K]$ so there is a $V \in \mathcal{V}$ with $f(x) \in V$. Then $x \in f^{-1}[V]$ and so $\mathcal{U}$ is an open cover of K. As K is compact, this implies $\mathcal{U}$ has a finite subset which covers K. That is there are $V_1, V_2, \ldots, V_n \in \mathcal{V}$ so that

$$K \subseteq f^{-1}[V_1] \cup f^{-1}[V_2] \cup \cdots \cup f^{-1}[V_n]$$

We now show $\mathcal{V}_0 = \{V_1, V_2, \ldots, V_n\}$ is a cover of $f[K]$ and thus that $\mathcal{V}$ has the required finite subcover of $f[K]$.

Let $y \in f[K]$, then by the definition of preimage, this implies there is a $x \in K$ with $f(x) = y$. As $\{f^{-1}[V_1], f^{-1}[V_2], \ldots, f^{-1}[V_n]\}$ covers K we have $x \in f^{-1}[V_j]$ for at least one j. Therefore $y = f(x) \in V_j$. As y was any point of $f[K]$ this implies $\mathcal{V}_0$ covers $f[K]$ as required. □

Theorem 3. *Let $A \subseteq \mathbb{R}$ be a compact subset of $\mathbb{R}$. Then A has a maximum and minimum. (Be sure you know the definition of maximum and minimum.)*

Proof. We know that compact subsets of $\mathbb{R}$ are closed and bounded. (In fact a compact subset of any metric space is closed and bounded.) Let $A \subseteq \mathbb{R}$ be compact. Then it is bounded and thus it has sup and a inf. Set

$$m = \inf(A), \qquad M = \sup(A).$$

We show that M is a maximum of A (the proof that m is a minimum is almost identical). As $M = \sup A$, we have that M is an adherent point of A. As A is closed this implies $M \in A$. Thus $M \in A$ and for all $a \in A$ we have $a \leq M$ by the definition of $M = \sup(A)$ as the least upper bound of A. Thus M is the maximum of A. □

Problem 1. Related to this proof here are a few proofs that I would find reasonable to ask on the final.

(a) Let A be a compact subset of the metric space E. Then is bounded in E. *Hint:* For this problem using the open cover definition of compactness is easiest. Choose any point $p \in E$. Then the set of open balls

$$\mathcal{U} = \{B(p, r) : r > 0\}$$

is an open cover of A. So there is a finite subcover

$$\mathcal{U}_0 = \{B(p, r_1), B(p, r_2), \ldots, B(p, r_n)\}$$

which covers A, therefore

$$A \subseteq \bigcup_{j=1}^{n} B(p, r_j) = B(p, r_{\max})$$

where $r_{\max} = \max(r_1, r_1, \dots, r_n)$. This shows A is bounded. □

(b) Let A be a compact subset of a metric space E. Then A is closed in E. *Hint:* This time it is easier to use sequential compactness. We just need to show that A contains all its adherent points. Let p be an adherent point of A. Then, by the definition of adherent point, for each positive integer n there is an point $a_n \in B(p, 1/n) \cap A$. Thus $a_n \in A$ and $d(a_n, p) < 1/n$. Thus $\lim_{n\to\infty} a_n = p$. (You can just say this is the case. The proof, which we have done several times before is for $\varepsilon > 0$ choose N with $1/N < \varepsilon$. Then $n \geq N$ implies $d(p, a_n) < 1/n < 1/N < \varepsilon$.) As A is sequentially compact the sequence $\langle a_n \rangle_{n=1}^{\infty}$ has a subsequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$ that converges to a point of A. That is $\lim_{k\to\infty} a_{n_k} = a$, where $a \in A$. But as this is a subsequence of the convergent sequence $\langle a_n \rangle_{n=1}^{\infty}$ it has the same limit,

$$a = \lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n = p.$$

Thus $p = a \in A$ and therefore A contains all its adherent points. □

(c) If $A \subseteq \mathbb{R}$ is bounded above, then $M = \sup(A)$ is an adherent point of A. *Hint:* Towards a contradiction, assume M is not an adherent point of A. Then there is a ball $B(M, r) = (M - r, M + r)$ with $B(M, r) \cap A = \varnothing$. If $x \in A$, then $x \leq M$ because M is an upper bound (in fact the least upper bound) of A. That is if $x \in A$, then $x \notin [M, \infty)$. But if $M - r < x \leq M$, that is $x \in (M - r, M]$ then $x \in B(M, r)$ and therefore $x \notin A$ as $B(M, r) \cap A = \varnothing$. But then if $x \in A$ we have $x \notin (M - r, M + r) \cup [M, \infty) = (M - r, \infty)$. Thus $x \leq M - r$, which contradicts that M is the least upper bound of A. □

3. Continuous functions and connected sets

Definition 4. A ***disconnection*** of the metric space E is two subsets $A, B \subseteq E$ such that

(i) $E = A \cup B$,
(ii) $A \cap B = \varnothing$.
(iii) $A \neq \varnothing$ and $B \neq \varnothing$,
(iv) A and B are open subsets of E.

If E has a disconnection, then it is ***disconnected***. □

A English equivalent is: E is disconnected if and only if it is the disjoint union of two nonempty open subsets of E. (Here "disjoint union" combines conditions (i) and (ii), "nonempty" is conditions (iii) and "open" is condition (iv).

Definition 5. A metric space is ***connected*** if and only if it it is not disconnected. □

Our one deep result about the existence of connected sets is

Theorem 6. *A subset of* $\mathbb{R}$ *is connected if and only if it is an interval.* □

In our *Continuous functions are wonderful* theorem, the condition that plays best with connected set is that the preimage of open sets are open. We have proven this before, but here it is again.

Theorem 7. *The continuous image of a connected set is connected. A little more precisely: If* E *is a connected metric space and* $f\colon E \to E'$ *is a continuous function, then the image* $f[E]$ *is a connected subset of* E'*.*

Proof. Towards a contradiction, assume $f[C]$ is not connected and let $f[C] = A \cup B$ be a disconnection of $f[C]$. Then A and B are open in $f[C]$ and as preimages of open sets are open $f^{-1}[A]$ and $f^{-1}[B]$ are open in E. By the definition of disconnection $A \neq \varnothing$ and $B \neq \varnothing$. Thus $f^{-1}[A] \neq \varnothing$ and $f^{-1}[B] \neq \varnothing$. By basic properties of preimages

$$f^{-1}[A] \cap f^{-1}[B] = f^{-1}[A \cap B] = f^{-1}[\varnothing] = \varnothing.$$

Thus

$$E = f^{-1}[A] \cup f^{-1}[B]$$

is a disconnection of E, contradicting that E is connected. □

Definition 8. An ***arc*** in a metric space, E, is a continuous function $f\colon [0,1] \to E$. □

Definition 9. A metric space E is ***arc-wise connected*** if and only if for all $p_0, p_1 \in E$, there is an arc $f\colon [0,1] \to E$ with $f(0) = p_0$ and $f(1) = p_1$. □

Theorem 10. *Every arcwise connected space is connected.*

Proof. Let E be arcwise connected and, towards a contradiction, assume E is not connected. Let $E = A \cup B$ be a disconnection and choose $p_0 \in A$ and $p_1 \in B$. (These points exist as $A, B \neq \varnothing$.) As E is arcwise connected there is a continuous map $f\colon [0,1] \to E$ with $f(0) = p_0$ and $f(1) = p_1$.

Let

$$A^* = f^{-1}[A] = \{t : f(t) \in A\}, \qquad B^* = f^{-1}[B] = \{t : ft) \in B\}.$$

As A and B are open and f continuous, the sets A^* and B^* are open (preimages of open sets are open strikes again). Then the argument used in the proof of Theorem 7 shows that $[0,1] = A^* \cup B^*$ is a disconnection of $[0,1]$, contradicting that the interval $[0,1]$ is connected. □