Mathematics 554 Homework.

There will be a quiz on Monday where you will have to

- Give the definition of a ***metric space*** (E, d) as on as on page 50 of the notes.
- The definition an ***open ball***, $B(p, r)$, and of a i***closed ball***, $\overline{B}(p, r)$, in a metric space.
- If (E, d) is a metric space know the definition of U being an ***open*** subset of E.

We have recently shown:

Proposition 1. *Any polynomial $p(x)$ on a bounded interval $[a, b]$ is Lipschitz.* □

And we also have

Theorem 2 (Lipschitz intermediate value theorem)**.** *Let $f\colon [a, b] \to \mathbb{R}$ be Lipschitz and assume $f(a)f(b) < 0$. Then there is a $\xi \in (a, b)$ with $f(\xi) = 0$.* □

The condition $f(a)f(b) < 0$ is just shorthand for saying that $f(a)$ and $f(b)$ have opposite signs, i.e. one is positive and one is negative. So the graph looks like one of the two in the following figure:

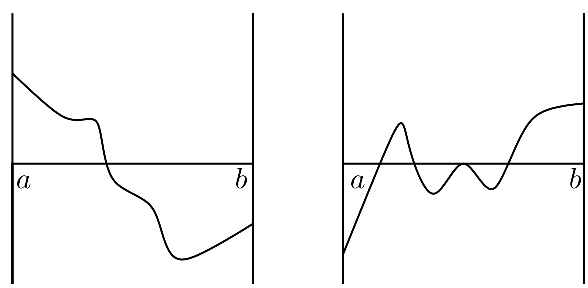

FIGURE 1. A Lipschitz function on $[a, b]$ whose values at the endpoint have opposite signs has a zero in the interval.

Just as review assume we are in the case that $f(a) > 0$ and $f(b) > 0$. Then to show the existence of ξ we set

$$S = \{x \in [a, b] : f(x) < 0\}.$$

This is a bounded subset of $\mathbb{R}$ and so by the Least Upper Bound axiom

$$\xi = \sup(S)$$

exists. Then it is "just" a matter of messing around with inequalities and that f is Lipschitz to show that

$$f(\xi) \leq 0, \quad \text{and} \quad f(\xi) \geq 0$$

both hold and therefore $f(\xi) = 0$.

Back to polynomials.

Problem 1. State an intermediate value theorem for polynomials on bounded intervals. □

Now let us show that all polynomials of degree 3 have a real root. That is we wish to show

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

has a solution where b_0, b_1, b_2, b_3 are constants and $b_3 \neq 0$. By dividing by $b_3 \neq 0$, this has a solution if and only if

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = 0$$

has a solution. So it is enough to show any polynomial equation of the form

$$x^3 + b_2x^2 + b_1x + b_0 = 0$$

has a zero. Set

$$p(x) = x^3 + b_2x^2 + b_1x + b_0.$$

When $x \neq 0$ this can be written as

$$p(x) = x^3\Big(1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3}\Big)$$

Problem 2. Show if $|x| \geq 1$, then

$$\frac{1}{|x|^3} \leq \frac{1}{|x|^2} \leq \frac{1}{|x|}.$$

Be explicit about where $|x| \geq 1$ is used. □

Problem 3. Show if $|x| \geq 1$, then

$$\left|\frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3}\right| \leq \frac{|b_2| + |b_1| + |b_0|}{|x|}.$$

□

Problem 4. Show if $|x| \geq \max\{1, 2(|b_2| + |b_1| + |b_0|)\}$, then

$$\left|\frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3}\right| \leq \frac{1}{2}.$$

□

Problem 5. Show if $|x| \geq \max\{1, 2(|b_2| + |b_1| + |b_0|)\}$, then

$$1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3} \geq \frac{1}{2}$$

and therefore

$$1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3}$$

is positive. *Hint:* Reverse triangle inequality. □

Problem 6. Let

$$b = \max\{1, 2(|b_2| + |b_1| + |b_0|)\}.$$

Show

$$p(b) > 0$$

and

$$p(-b) < 0$$

Hint: Recall

$$p(x) = x^3\Big(1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3}\Big)$$

and note $b^3 > 0$ and $(-b)^3 = -b^3 < 0$. Then problem 5 should let you finish the proof. □

Problem 7. Prove that $p(x) = 0$ has a root in the interval $(-b, b)$. Thus every cubic polynomial has a real root. □

Remark. It is not hard to modify the argument we have just given to show every polynomial of odd degree has at least one real root. □

Problem 8. In *Notes on Analysis* do problems 3.1 and 3.2 on pages 50 and 51.