Mathematics 554 Homework.

Problem 1. In *Notes on analysis* do problem 3.14.

Let (E, d) be a metric space and $f\colon E \to \mathbb{R}$ be a function. Then f is ***Lipschitz*** if and only if there is a constant $M > 0$ such that for all $p, q \in E$

$$|f(p) - f(q)| \le M d(p, q).$$

In the notes we have

Theorem 1. *If $f\colon E \to \mathbb{R}$ is Lipschitz, then for any $c \in \mathbb{R}$ the following are open sets*

$$f^{-1}[(-\infty, c)] := \{p \in E : f(p) < c\}$$
$$f^{-1}[(c, \infty)] := \{p \in E : f(p) > c\}$$

Problem 2. As more practice in both working open set and inequalities prove $U = \{p \in E : f(p) > c\}$ is open. *Hint:* Let $p \in U$. We need to find $r > 0$ such that $B(p, r) \subseteq U$. By definition $f(p) > c$. Let M be the Lipschitz constant for f and set

$$r = \frac{f(p) - c}{M}.$$

Then $r > 0$ as $f(p) > c$. Let $x \in B(p, r)$. Then $d(p, x) < r$. Now use the Lipschitz property $|f(p) - f(x)| \le M d(p, x)$ and the adding and subtracting trick

$$f(x) = f(p) + (f(x) - f(p)) \ge f(p) - |f(x) - f(p)|$$

to show $f(x) > c$ and thus $x \in U$. As x was an arbitrary element of $B(p, r)$ this shows $B(p, r) \subseteq U$ and completes the proof that U is open. □

Problem 3. Let (E, d) be a metric space and let $p_0 \in E$. Define a function $f\colon E \to \mathbb{R}$ by

$$f(p) = d(p, p_0).$$

Show for all $p, q \in E$ that

$$|f(p) - f(q)| \le d(p, q).$$

Hint: Reverse triangle inequality. □

Definition 2. Let E be a metric space and $\langle p_n \rangle_{n=1}^{\infty} = \langle p_1, p_2, p_3, \ldots \rangle$ a sequence in E. Then

$$\lim_{n\to\infty} p_n = p$$

if and only if for all $\varepsilon > 0$ there is $N > 0$ such that

$$n > N \implies d(p_n, p) < \varepsilon.$$

In this case we say the sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges to p. □

Problem 4. Have this definition memorized. There will be a quiz on it when we get back from the break. □