Mathematics 554 Homework.

Definition 1. Let $\langle p_n\rangle_{n=1}^{\infty}$ be a sequence in a metric space E. Then the sequence is bounded if and only if there is a point $p \in E$ and a real number $R > 0$ such that $d(p, p_n) \le R$ for all n. □

Problem 1. Prove that a convergent sequence in a metric space is bounded. *Hint:* Let $\langle p_n\rangle_{n=1}^{\infty}$ be convergent, say $\lim_{n\to\infty} p_n = p$. Let $\varepsilon = 1$ in the definition of convergence to get a N such that

$$n \le N \quad \text{implies} \quad d(p, p_n) < 1.$$

Now prove if

$$R = \max\{1, d(p, p_1), d(p, p_2), \cdots, d(p, p_N)\}$$

then $d(p, p_n) \le R$ for all n. □

We have done the next problem in class, but it is worth repeating.

Problem 2. Let $f\colon E \to E'$ be a Lipschitz map between metric spaces. Let $\langle p_n\rangle_{n=1}^{\infty}$ be a convergent sequence in E, say $\lim_{n\to\infty} p_n = p$. Then

$$\lim_{n\to\infty} f(p_n) = f(p)$$

in E'. *Hint:* By the definition of Lipschitz there is $M > 0$ such that

$$d'(f(x), f(y)) \le M d(x, y)$$

for all $x, y \in E$. In particular this implies

$$d'(f(p_n), f(p)) \le M d(p_n, p).$$

You should be able to use this and the definition of $p_n \to p$ to find a N so that $n \ge N \implies d'(f(p_n), f(p)) < \varepsilon$. □

Theorem 2. *Let E be a metric space and $A \subseteq E$ a closed subset of E. Let $\langle a_n\rangle_{n=1}^{\infty}$ be a sequcence with $a_n \in A$ for all n. Assume $\lim_{n\to\infty} a_n = p$. Then $p \in A$.*

A nice restatement is that a closed set contains the limits of all its convergent sequences.

Problem 3. Prove the theorem. *Hint:* Towards a contradiction assume that $p \notin A$. Then $p \in \mathcal{C}(A)$ (the complement of A in E) is open. Thus there is a $r > 0$ such that $B(p, r) \subseteq \mathcal{C}(A)$. That is $B(p, r) \cap A = \varnothing$. Letting $\varepsilon = r$ in the definition of $p_a \to p$ give a N such that $n \ge N \implies d(a_n, p) < r$. Show this leads to a contradiction. □

Definition 3. Let A be a metric space and $A \subseteq E$ a subset of E. Then $p \in E$ is an ***adherent point*** of A if and only if for all $r > 0$ we have $B(p, r) \cap A \ne \varnothing$. □

Here is an equivalent definition which may be easier to use.

Definition 4. Let A be a metric space and $A \subseteq E$ a subset of E. Then $p \in E$ is an ***adherent point*** of A if and only if for all $r > 0$ there is $a \in A$ with $d(p, a) < r$. □

Proposition 5. *If p is an adherent point of A, then there is a sequence $\langle a_n \rangle_{n=1}^{\infty}$ with $a_n \in A$ for all n and $\lim_{n\to\infty} a_n = p$.*

Problem 4. Prove this. *Hint:* Let n be a positive integer. Explain why the definition of adherent point implies there is an $a_n \in A$ such that $d(a_n, p) < 1/n$. Then show $\lim_{n\to\infty} a_n = p$. □