Mathematics 554 Test 1 Name:

You are to use your own calculator, no sharing.
Show your work to get credit.

1. (10 points) What is the sum of the series $S = \sum_{k=0}^{29} P_0(1+r)^k$?

$S =$ $\boxed{\dfrac{P_0 - P_0(1+r)^{30}}{-r}}$

$\dfrac{\text{first} - \text{next}}{1 - \text{ratio}}$

$S = \dfrac{P_0(1+r)^0 - P_0(1+r)^{30}}{1-(1+r)} = \dfrac{P_0 - P_0(1+r)^{30}}{-r}$

$\dfrac{P_0(1+r)^2}{P_0(1+r)}, = (1+r) = \text{ratio}$

2. (15 points) (a) Let $S \subseteq \mathbb{R}$. Define what it means for S to be bounded above.

S is bounded above if and only if there exists a $b \in \mathbb{R}$ such that for all $x \in S$, $x \leq b$.

(b) State what it means for b to be the ***least upper bound*** for the set $S \subseteq \mathbb{R}$.

b is a least upper bound for ~~the~~ $S \subseteq \mathbb{R}$ iff S is nonempty, b is an upper bound for S, and for any upper bound c of S, $b \leq c$.

(c) State the ***least upper bound axiom***.

If $S \subseteq \mathbb{R}$ is bounded above, then there exists a least upper bound of S.

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3. (15 points) (a) State the ***binomial theorem***.

For positive integer n and real numbers x, y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

(b) Simplify $\frac{(a+h)^3 - 2a^3 + (a-h)^3}{h^2}$. (The answer should have no h in the denominator.)

The simplification is $6a$

$$\frac{a^3 + 3a^2h + 3ah^2 + h^3 - 2a^3 + a^3 - 3a^2h + 3ah^2 - h^3}{h^2}$$

$$= \frac{6ah^2}{h^2} = 6a$$

4. (10 points) Give examples of (no proofs required).

(a) A subset of $\mathbb{R}$ that is bounded below, but not bounded above.

$(0, \infty)$

(b) A subset of $\mathbb{R}$ that is bounded below, but has no minimum.

$(0, 1)$

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5. (10 points) Use the least upper bound axiom to show the set $S = \{2^n : n \in \mathbb{N}\}$ is not bounded above.

Proof. Towards a contradiction assume the set S is bounded above. By the least upper bound axiom, a least upper bound of S exists. Let $b = \sup(S)$. Since $n \in \mathbb{N}$, we have $n+1 \in \mathbb{N}$. So

$$2^{n+1} \leq b \quad [\text{for } n \in \mathbb{N}$$

$$2^n \leq \frac{b}{2} < b \quad [\text{for } n \in \mathbb{N}]$$

We have found an upper bound less than the least upper bound which is a contradiction. Therefore, $S = \{2^n : n \in \mathbb{N}\}$ is not bounded above. $\square$

Very nicely done

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$x^2+2xy+2y^2$

6. (10 points) Prove $x^2+2y+2y^2 \geq 0$ with equality if and only if $x=y=0$.

Proof. We have

$$x^2+2xy+2y^2 = x^2+2xy+y^2+y^2$$
$$= (x+y)^2+y^2 \geq 0.$$

This is because sums of squares are always non-negative. Therefore, equality only holds when

$$x+y=0 \quad \text{and} \quad y=0.$$

Plugging $y=0$ into $x+y=0$ gives $x=0$. Hence equality only holds if and only if $x=y=0$. □

Good job

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7. (15 points) (a) Define what if means for a function $f\colon [a,b] \to \mathbb{R}$ to be ***Lipschitz***.

A function $f : [a,b] \to \mathbb{R}$ is Lipschitz if and only if there exists a constant M such that

$$|f(x) - f(y)| \leq M|x-y|$$

for all $x, y \in [a,b]$.

(b) Show the function $f(x) = 3x^2 - x + 17$ is Lipschitz on the interval $[-4, 4]$.

Proof. By algebra,

$$
\begin{aligned}
|3x^2 - x + 17 - 3y^2 + y - 17| &= |3x^2 - x - 3y^2 + y| \\
&= |3x^2 - 3y^2 - x + y| \\
&= |3(x^2 - y^2) - (x - y)| \\
&= |3(x+y)(x-y) - (x-y)| \\
&= |(x-y)(3(x+y) - 1)|. \\
&= |x-y||3x + 3y - 1| \\
&\leq |x-y|(|3x| + |3y| + |-1|) \quad (1)
\end{aligned}
$$

Good

Since $x, y \in [-4, 4]$, we know that $|x| \leq 4$ and $|y| \leq 4$.
So by (1), we have

$$
\begin{aligned}
|3x^2 - x + 17 - 3y^2 + y - 17| &\leq |x-y|(3 \cdot 4 + 3 \cdot 4 + 1) \\
&\leq 25|x-y|.
\end{aligned}
$$

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8. (10 points) Let $f: [a, b] \to \mathbb{R}$ satisfy

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$$

where $M > 0$. Let $x_0 \in [a, b]$ be a point where f is "approximately nonnegative" in the sense that there are near x_0 where $f \geq 0$. More precisely assume that for all $\delta > 0$ there is $x \in [a, b]$ with

$$|x - x_0| < \delta \quad \text{and} \quad f(x) \geq 0.$$

Show $f(x_0) \geq 0$.

Towards a contradiction, assume $f(x_0) < 0$. Then set $\delta = \frac{|f(x_0)|}{2M}$. Then there exists a $x \in [a,b]$ such that $|x - x_0| < \delta$ and $f(x) \geq 0$. Then,

$$\begin{aligned}
0 \leq f(x) &= f(x_0) + (f(x) - f(x_0)) && \text{add and subtract trick} \\
&\leq f(x_0) + |f(x) - f(x_0)| && \text{Since } f(x) - f(x_0) \leq |f(x) - f(x_0)| \\
&\leq f(x_0) + M|x - x_0| && \text{as } |f(x) - f(x_0)| \leq M|x - x_0| \\
&< f(x_0) + M\delta && \text{as } |x - x_0| < \delta \\
&= f(x_0) + M\frac{|f(x_0)|}{2M} && \text{as } \delta = \frac{|f(x_0)|}{2M} \\
&= f(x_0) + \frac{1}{2}|f(x_0)| && \text{algebra} \\
&= f(x_0) - \frac{1}{2}f(x_0) && |f(x_0)| = -f(x_0) \text{ as } f(x_0) < 0. \\
&= \frac{1}{2}f(x_0) && \text{algebra.} \\
&< 0 && \text{as } f(x_0) < 0.
\end{aligned}$$

So we have $0 < 0$, a contradiction.

Good

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