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Mathematics 554 Test 2

Name:__

1. (15 points) (a) Let $\langle p_n \rangle_{n=1}^{\infty}$ be sequence in the metric space E and $p \in E$. Define $\lim_{n\to\infty} p_n = p$.

then $\lim_{n\to\infty} P_n = P$ iff for all $\varepsilon > 0$ there is N such that $n \geq N$ implies $d(P_n, P) < \varepsilon$

we can say that this sequence converges to P. #

15/15

(b) Let E and E' be metric spaces, $f: E \to E'$ and $p_0 \in E$. Define what it means for f to be ***continuous*** at p_0.

then f is continuous at P_0 iff for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $P \in E$

$d(P, P_0) < \delta$ implies $d'(f(p), f(P_0)) < \varepsilon$ #

Good Job

(c) Let $f: E \to E'$ be a map between metric spaces, $p_0 \in E$, and $q_0 \in E'$. Define $\lim_{p\to p_0} f(p) = q_0$.

then $\lim_{P \to P_0} f(p) = q_0$ iff for all $\varepsilon > 0$ ~~there is~~ $\delta > 0$ such that for all $P \in E$.

$0 < d(P, P_0) < \delta$ implies $d'(f(p), q_0) < \varepsilon$

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2. (5 points) Let E be a metric space, $A \subseteq E$ a subset of E, and $p \in E$. Define what it means for p to be an ***adherent point*** of A.

P is an adherent point of A iff for every $r > 0$ we have $A \cap B(p, r) \neq \emptyset$ #

20

3. (20 points) Let E be a metric space and $U \subseteq E$.

(a) Define what it means for U to be ***open***.

$\forall_{x \in U} \exists_{r>0} \; B(x,r) \subseteq U$

(b) Prove: If U and V are open, then so is $U \cap V$.

Let U, V open.

Assume $x \in U \cap V$. Then $x \in U$ and $x \in V$. Then $\exists_{r_1 > 0} \; B(x, r_1) \subseteq U$ and $\exists_{r_2 > 0} \; B(x, r_2) \subseteq V$.

Then $B(x, \min(r_1, r_2)) = \{y : d(y,x) < r_1 \text{ and } d(y,x) < r_2\} \subseteq B(x,r_1) \cap B(x,r_2) \subseteq U \cap V$

Then $U \cap V$ is open.

(c) Let $U_1, U_2, U_3, \ldots$ be open sets in E. Prove the union $\bigcup_{n=1}^{\infty} U_n$ is also open.

Let $x \in \bigcup_{n=1}^{\infty} U_n$. Then $x \in U_j$ for some $j \in \{1, 2, \ldots\}$ and U_j is open.

Then $\exists_{r>0} \; B(x,r) \subseteq U_j$. Then $B(x,r) \subseteq \bigcup_{n=1}^{\infty} U_n$, so $\bigcup_{n=1}^{\infty} U_n$ is open.

(d) Give an example of open sets $U_1, U_2, U_3, \ldots$ in $\mathbb{R}$ such that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not open (no proof required).

$U_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$ for all $n \in \{1, 2, 3, \ldots\}$

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4. (15 points) Let $f\colon E \to E'$ be a function such that for some $M > 0$ and all $p, q \in E$ the inequality

$$d'(f(p), f(q)) \leq Md(p, q)$$

holds. (That is, in our terminology, f is ***Lipschitz***). Let $V \subseteq E'$ be an open subset of E'. Show the set

$$U := \{p \in E : f(p) \in V\}$$

is open in E.

Proof. Let $f: E \to E'$ be a function such that for some $M>0$ and all $p, q \in E$, the inequality

$$d'(f(p), f(q)) \leq Md(p,q) \text{ holds.}$$

Let $V \subseteq E'$ be an open set of E'. We will show $U := \{p \in E : f(p) \in V\}$ is open in E.

Let $p \in U$. By definition of U, $f(p) \in V$. Since V is open, there exists an $r>0$ such that $B(f(p), r) := \{q \in V : d'(f(p), q) < r\} \subseteq V$.

Let $\rho = \frac{r}{M} > 0$. Let $q \in B(p, \rho)$, so $d(p,q) < \rho$. We have

$$\begin{aligned} d'(f(p), f(q)) &\leq M d(p,q) \\ &< M\rho \\ &= M\frac{r}{M} \\ &= r. \end{aligned}$$

Since $d'(f(p), f(q)) < r$, $f(q) \in B(f(p), r)$ which implies $q \in U$. Since $q \in B(p, \rho)$ implies $q \in U$, $B(p, \rho) \subseteq U$ so U is open in E. $\square$

15

5. (15 points) (a) State the intermediate value theorem for Lipschitz function $f : [a, b] \to R$.

Let $f : [a,b] \to \mathbb{R}$ be Lip on $[a,b]$ + $f(a)f(b) < 0$

then $\exists \xi \in [a,b]$ st $f(\xi) = 0$

(b) Prove: For any $a > 0$ the polynomial $p(x) = x^6 + x^5 + x - a$ has a root in the interval $(0, \infty)$. (You may assume that polynomials on bounded intervals are Lipschitz.)

$p(x) = x^6 + x^5 + x - a$ w/ $a > 0$

then $p(0) = -a < 0$ + $p(a) = a^6 + a^5 > 0$

so $f(0)f(a) < 0$. As $p(x)$ is a polynomial, it is

Lip on $[0,a]$. Thus, By the Lip IVT,

$\exists \xi \in [0,a]$ st $f(\xi) = 0$

good

15

6. (15 points) Let $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ be sequences of real numbers with

$$\lim_{n\to\infty} x_n = x, \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

Use the definition of limit to prove:

$$\lim_{n\to\infty} (13x_n - 9y_n) = 13x - 9y.$$

Let $\varepsilon > 0$. Then,

$$\star \ |(13x_n - 9y_n) - (13x - 9y)| = |(13x_n - 13x) + (9y - 9y_n)|$$
$$\leq |13x_n - 13x| + |9y - 9y_n|$$

As $x_n \to x$, there is N_1 such that $n \geq N_1 \Rightarrow \underbrace{d(x_n, x)}_{=|x_n - x|} < \frac{\varepsilon}{2(13)}$

As $y_n \to y$, there is N_2 such that $n \geq N_2 \Rightarrow \underbrace{d(y_n, y)}_{=|y_n - y|} < \frac{\varepsilon}{2(9)}$

Let $N = \max\{N_1, N_2\}$. Then by the inequality, $\star$,

$$|(13x_n - 9y_n) - (13x - 9y)| \leq 13|x_n - x| + 9|y - y_n|$$
$$< 13\frac{\varepsilon}{2(13)} + 9\frac{\varepsilon}{2(9)}$$
$$= \varepsilon$$

So we have $d(13x_n - 9y_n, 13x - 9y) < \varepsilon$, thus $\lim_{n\to\infty} (13x_n - 9y_n) = 13x - 9y$

15

7. (15 points) (a) State the ***Cauchy-Schwartz inequality***.

$$\vec{a} \circ \vec{b} \le \|\vec{a}\| \, \|\vec{b}\|$$

(b) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = 2x - 3y + 5z.$$

Show f is Lipschitz.

$$f(x, y, z) = (2, -3, 5) \circ (x, y, z)$$

$$|f(x_1, y_1, z_1) - f(x_2, y_2, z_2)| =$$

$$= |(2, -3, 5) \circ (x_1, y_1, z_1) - (2, -3, 5) \circ (x_2, y_2, z_2)|$$

$$= |(2, -3, 5) \circ (x_1 - x_2, y_1 - y_2, z_1 - z_2)|$$

$$\le \|(2, -3, 5)\| \, \|x_1 - x_2, y_1 - y_2, z_1 - z_2\|$$

$$= M \|x_1 - x_2, y_1 - y_2, z_1 - z_2\|$$

$L = d((x_1, y_1, z_1), (x_2, y_2, z_2))$

where $M = \|(2, -3, 5)\|$. $\square$

(For what it matters

$$M = \sqrt{2^2 + (-3)^2 + 5^2}$$
$$= \sqrt{4 + 9 + 25}$$
$$= \sqrt{38}$$

12