SESSION 1 PROBLEMS

(a) Let
$$E = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant 1\}$$
be a full sphere. Suppose that $P \subset E$ is such that $|P| = 5$. Prove that you can cut the full sphere E into two perfect halves in such a way that one half contains at least 4 points of P. (For the purpose of this exercice, a point that is directly on the cut is considered to be in both halves of the sphere)

(b) Show that 2023 cannot be written as the sum of two integer squares.

(c) Let T be a triangle in the cartesian plane, and let T' be its reflexion across the y-axis. Prove that, with at most three straight cuts, you can cut T into pieces in such a way that these pieces can be reassembled into T' using only translations and rotations of the pieces.

(d) Supposing that the expression
$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}$$
converges, find its value.

(e) Let $P(x)$ be a polynomial of degree 2024 satisfying $P(k) = k$ for $k = 1, \dots, 2024$, and $P(0) = 1$. Find $P(-1)$.

(f) Consider a chess board. Suppose that you have 2 by 1 tiles that fit perfectly onto two adjacent squares of the chess board. Prove that it is possible to cover the whole chess board with 32 of these tiles.

Suppose now that we remove two opposite corners from the chess board. Is it possible to perfectly tile the 62 remaining squares using 31 of these tiles?