Session 2 Problems: Calculus

Notation:

- $\mathbb{R}$ is the set of real numbers i.e. the number line.
- $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ is the set of complex numbers.

Tools that may be useful:

- Intermediate Value Theorem:
  Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f(a) = c < f(b) = d$. Then for all $y \in [c, d]$, there exists $x \in [a, b]$ such that $f(x) = y$.
- Mean Value Theorem:
  For a differentiable function $f : [a, b] \to \mathbb{R}$ there exists an $x \in (a, b)$ so that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

Problems:

1. Let $f : [a, b] \to [a, b]$ be a continuous function. Show that f has a fixed point; i.e. show that there is a $c \in [a, b]$ with $f(c) = c$.
2. Show that not all the zeros of the polynomial $P(x) = x^4 - \sqrt{7}x^3 + 4x^2 - \sqrt{22}x + 15$ are real.
3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying:
   $$|f(x) - f(y)| \leq |x - y|^2$$
   for all $x, y \in \mathbb{R}$.
4. Compute
   $$S = \int \frac{\sin(x)}{\sin(x) + \cos(x)}\, dx$$
5. Compute
   $$\int_0^{\pi/2} \ln(\sin x) dx$$
6. Determine
   $$\max_{z \in \mathbb{C},\ |z|=1} |z^3 - z + 2|.$$