SESSION 3 PROBLEMS: SEQUENCES AND SERIES

(a) Determine if the series

$$\sum_{n=0}^{\infty} \frac{n^n}{2^{n^2}}$$

converges or diverges.

(b) Suppose $k = 2024/745$. Determine if the series

$$\sum_{n=1}^{\infty} \frac{n! k^n}{(n+1)^n}$$

converges or diverges.

(c) Determine if the series

$$\sum_{n=3}^{\infty} \frac{1}{n \log(n) \log\log(n)}$$

converges or diverges. ($\log$ denotes the natural logarithm)

(d) Suppose a_n is a convergent sequence, and that $a_n \to \ell$. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} a_n = \ell.$$

(e) Suppose that the sequence a_n is monotone and that

$$\sum_{n=1}^{\infty} a_n$$

converges. Show that

$$\sum_{n=1}^{\infty} n(a_n - a_{n+1})$$

converges.

(f) Let a_n be a sequence of positive reals numbers that satisfy $a_n \leqslant a_{2n} + a_{2n+1}$ for all $n \in \mathbb{N}$. Prove that

$$\sum_{n=1}^{\infty} a_n$$

diverges.

(g) Let a_n be a sequence of real numbers satisfying $a_n = \sum_{k=n+1}^{\infty} a_k^2$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_n = 0$ for all $n \in \mathbb{N}$.

(h) Prove that there exists some constant $C \in \mathbb{R}$ such that $|\sum_{n=1}^{N} \cos(n)| < C$ and $|\sum_{n=1}^{N} \sin(n)| < C$ for all $N \in \mathbb{N}$.