SESSION 4 PROBLEMS: LINEAR ALGEBRA

Tools:

(a) Let A, B, C, D be 4 points in the plane that form a quadrilateral. Prove that the midpoints of AB, BC, CD and DA form a parallelogram.

(b) Let A, B, C, D be 4 points in the plane. Prove that $|AB||CD| + |BC||AD| \geqslant |AC||BD|$.

(c) Does there exist $n \times n$ matrices A, B with $AB - BA = I_n$.

(d) Let F_n be the nth Fibonacci number. Use the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ to show that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

(e) For arbitrary value $x_1, x_2, \ldots, x_n$ find the determinant of

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

(f) Let A be an $n \times n$ symmetric invertible matrix with positive real entries, $n \geqslant 2$. Show that A^{-1} has at most $n^2 - 2n$ entries equal to zero.

(g) For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?