Mathematics 552
Take Home Portion of Final

This is due in at the time of the in class portion of the final, Monday, May 1, 9:00 AM.

As you have several days to do the exam I expect the results of be well written and complete. ***Complete means more English, not necessarily more algebra.*** It will not bother me if you say "it now follows by a calculation that" and skip a little of the algebra. (For example something like "Then we have $\lambda^2 - 2\lambda + 2 = 0$ and so by the quadratic formula it follows that $\lambda = 1 \pm i$" is fine.) But you should explain what you are computing and what it has to do with the solution to the problem. One good way to do this is to exchange first drafts of a problem with someone else that has done the problem and have them make comments on where they had trouble reading your solution while you make a similar comments on their paper. You don't have to end up agreeing what is the "best" way to write things, but if someone is having trouble reading what you have written then it probably means that you needed to do some rewriting.

1. (10 Points) Find the real and imaginary parts of $\sec(x + iy)$.
2. (10 Points) Let n be a positive integer and a, b any two complex numbers. Show that $\max_{|z|\leq 2} |az^n + b| = 2^n|a| + |b|$.
3. (10 Points) The ***cross ratio*** of four distinct complex numbers z_0, z_1, z_2, z_3 is defined to be
$$[z_0, z_1, z_2, z_3] = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_3)(z_2 - z_1)}.$$
Let a, b, c, d be complex numbers with $ad - bc \neq 0$ and define
$$T(z) = \frac{az + b}{cz + d}.$$
(A function of this form is called a ***Möbius transformation*** or a ***linear fractional transformation***.) Show that a Möbius transformation preserves cross ratios in the sense that
$$[T(z_0), T(z_1), T(z_2), T(z_3)] = [z_0, z_1, z_2, z_3].$$
4. (10 Points) Let $T(z) = \dfrac{az + b}{cz + d}$ where a, b, c, d are complex numbers with $ad - bc \neq 0$ be a Möbius transformation. Let $C_r(z_0)$ be the circle of radius r about z_0. Show that the image of $C_r(z_0)$ by T is either a circle or a straight line. (Recall that the image of $C_r(z_0)$ by T is the set $T[C_r(z_0)] = \{T(z) : z \in C_r(z_0)\}$.)

5. (10 Points) Let D be a domain and let f be an analytic function defined on D with $f \neq 0$ in D. Assume there is a function h so that

$$h'(z) = \frac{f'(z)}{f(z)}.$$

Show that $f(z) = ce^{h(z)}$ for some constant c. **Hint:** Note that $f(z) = ce^{h(z)}$ if and only if $f(z)e^{-h(z)} = c$ so it is enough to show that $f(z)e^{-h(z)}$ is constant. (You should supply the details why this is true.) But to show an analytic function is constant in a domain it is enough to show that its derivative is zero. So compute

$$\frac{d}{dz}\left(f(z)e^{-h(z)}\right) = f'(z)e^{-h(z)} + f(z)(-h'(z))e^{-h(z)}$$

and use $h'(z) = f'(z)/f(z)$.

6. (10 Points) Let D be a simply connected domain and let $f(z)$ be an analytic function defined on D so that $f(z) \neq 0$ in D. Then show that there is an analytic function $g(z)$ so that

$$e^{g(z)} = f(z).$$

(This can be restated as: Every non-vanishing analytic function on a simply connected domain has a logarithm.) **Hint:** This result is false if the domain is not simply connected so if you do not use that fact somewhere in your argument then you will lose points. Here is how to get started. As $f(z)$ is non-vanishing in D the function $f'(z)/f(z)$ is analytic in D and therefore it has an antiderivative in D (why?). Let $h(z)$ be such an antiderivative, so that $h'(z) = f'(z)/f(z)$. Then by the last problem we have $f(z) = ce^{h(z)}$ for some constant c. Let $g(z) = h(z) + a$ where a is a constant and see if you can choose a so that $e^{g(z)} = f(z)$.

7. (5 Points) This is an application of the last problem. Let D be a simply connected domain and let $f(z)$ be an analytic function defined on D so that $f(z) \neq 0$ in D. Then show there is an analytic function $q(z)$ defined on D so that $q(z)^2 = f(z)$. That is $f(z)$ has a square root. **Hint:** Don't make this hard. By the last problem there is a $g(z)$ with $f(z) = e^{g(z)}$. Now let $q(z) = e^{\frac{1}{2}g(z)}$. **Extra Credit:** As a variant on this make precise and prove the following: If $f(z)$ is a non-vanishing analytic function on a simply connected domain then $f(z)$ has an n root for any positive integer n.

8. (10 Points) Use the Residue Theorem to compute the following integrals.

(a) $\displaystyle\oint_{|z|=5} \frac{z}{z^2-9}\,dz,$ (b) $\displaystyle\oint_{|z|=4} \frac{e^z}{z^2+2z-15}\,dz.$

What to expect on the in class part of the final.

The in class part of the final will be, apart from various and sundry surprise mystery questions, mostly calculations similar to ones that have appeared on earlier exams and quizzes. You should also know the statements of the major theorems we have covered and be able to define the important terms. In terms of topics covered since the third exam, know the definition of a residue and the statement of the Residue Theorem.