Mathematics 552 Test #3

This is due in class on Monday 17 April, 2000. You may talk with each other about the test, but have to write the answers on your own. As it is a take home test the answers should be neatly written and be complete.

1. (25 points) Let $f(z) = \dfrac{e^z}{z^3}$.
   (a) Compute $\displaystyle\oint_{|z|=1} f(z)\,dz$.
   (b) Does $f(z)$ have an antiderivative in the domain $D = \{z \in \mathbb{C} : z \neq 0\}$? Justify your answer.
2. (15 points) If D is a bounded domain in $\mathbb{C}$ with area A. Then the ***center of mass*** of D is the point with coordinates $(\widetilde{x}, \widetilde{y})$ where
$$\widetilde{x} = \frac{1}{A}\iint_D x\,dxdy, \qquad \widetilde{y} = \frac{1}{A}\iint_D y\,dxdy.$$
   Let Γ be the boundary of D traversed in the positive direction, that moving so that D is always on the left. Then show
$$\int_\Gamma \overline{z}^2\,dz = 4A(\widetilde{y} + i\,\widetilde{x}).$$
3. (25) points)Prove the following:

   Theorem: *Let D be a simply connected domain and let φ be a harmonic function in D. Then there is an analytic function f in D so that* $\mathrm{Re}(f) = \varphi$.

   by completing the following steps.
   (a) Show that the function $g = \varphi_x - i\,\varphi_y$ is analytic in D.
   (b) Show that g has an antiderivative f in g. (That is there is an $f(z)$ in D with $f'(z) = g(z)$.)
   (c) Finish by showing it is possible to choose the antiderivative f so that $\mathrm{Re}(f) = \varphi$.
4. (15 points) Use Cauchy's integral formula to show that if $f(z)$ is analytic in $|z - z_0| \leq r$, then
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\,t})\,dt.$$
5. (20 points) Let $f(z)$ be an entire function that satisfies $|f(z)| \leq 100 + |z|$. Then show that $f(z)$ is linear. That is there are constants $a, b \in \mathbb{C}$ so that $f(z) = az + b$. Hint: A function is linear if and only if $f''(z_0) = 0$ for all z_0. Use that $f(z_0) = \frac{2!}{2\pi i}\oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^3}\,dz$ and proceed as in the proof of Liouville's theorem.