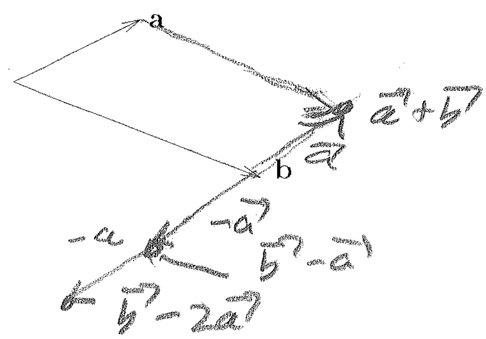
Mathematics 241 Test #1 Name: Key

Show your work to get credit. An answer with no work will not get credit.

1. (5 points) In the following figure draw and **label** the vectors $\mathbf{a}+\mathbf{b}$, $\mathbf{b}-\mathbf{a}$, and $\mathbf{b}-2\mathbf{a}$,



2. (7 points) Find the center and radius of the sphere with equation

$$x^2+y^2+z^2-4x+10y+6z=-8$$

Complete the square

Center = $(2,-5,-3)$

Radius = $\sqrt{30}$

$x^2-4x+4+y^2+10y+25+z^2+6z+9=-8+4+25+9$

$(x-2)^2+(y+5)^2+(z+3)^2=30=(\sqrt{30})^2$

3. (8 points) What is the area of the triangle with vertices's $(1,2,3)$, $(0,1,2)$, $(0,0,1)$?

If $P=(1,2,3)$, $Q=(0,1,2)$
$R=(0,0,1)$

Area = $\frac{\sqrt{2}}{2}$

Then $A=\frac{1}{2}|\vec{RP}\times\vec{RQ}|$

$\vec{RP}\times\vec{RQ}=\langle 1,2,2\rangle\times\langle 0,1,1\rangle=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2&2\\0&1&1\end{vmatrix}$

$=\left\langle \begin{vmatrix}2&2\\1&1\end{vmatrix}, -\begin{vmatrix}1&2\\0&1\end{vmatrix}, \begin{vmatrix}1&2\\0&1\end{vmatrix}\right\rangle=\langle 0,-1,1\rangle=-\vec{j}+\vec{k}$

So Area $=\frac{1}{2}\sqrt{(-1)^2+(1)^2}=\frac{\sqrt{2}}{2}$

4. (30 points) If $\mathbf{a} = \langle 3, 2, 1\rangle$, $\mathbf{b} = \langle 2, -2, -1\rangle$, and $\mathbf{c} = \langle 4, 2, 1\rangle$ then compute the following.

(a) $2\mathbf{a} + \mathbf{b}$ $= \langle 6, 4, 2\rangle + \langle 2, -2, -1\rangle$
$= \langle 8, 2, 1\rangle$

$\langle 8, 2, 1\rangle$

(b) The unit vector in the direction of $\mathbf{a}$

$$= \frac{\vec{a}}{|\vec{a}|} = \frac{\langle 3,2,1\rangle}{\sqrt{9+4+1}} = \frac{\langle 3,2,1\rangle}{\sqrt{14}}$$

$\frac{\langle 3,2,1\rangle}{\sqrt{14}} = \left\langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right\rangle$

(c) $\mathbf{a} \cdot \mathbf{b}$ $= \langle 3,2,1\rangle \cdot \langle 2,-2,-1\rangle$
$= 6 - 4 - 1 = 1$

1

(d) $\mathbf{b} \times \mathbf{c}$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & -1 \\ 4 & 2 & 1 \end{vmatrix}$

$$= \left\langle \begin{vmatrix} -2 & -1 \\ 2 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} 2 & -2 \\ 4 & 2 \end{vmatrix} \right\rangle = \langle 0, -(2+4), 4+8\rangle$$

$\langle 0, -6, 12\rangle$

(e) $|\mathbf{b}|$ $= \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

From (d)

(f) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ $= \langle 3,2,1\rangle \cdot \langle 0,-6,12\rangle$
$= 0 - 12 + 12 = 0$

0

(g) $\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{c}$.

$\vec{a} \cdot \vec{c} = |\vec{a}|\,|\vec{c}| \cos\theta$
$12 + 4 + 1 = \sqrt{9+4+1}\,\sqrt{16+4+1} \cos\theta$
$17 = \sqrt{14}\,\sqrt{21} \cos\theta$

$\cos\theta = \frac{17}{\sqrt{14}\sqrt{21}} = \frac{17}{7\sqrt{6}}$

(h) the component of $\mathbf{c}$ in the direction of $\mathbf{a}$

$$\mathrm{comp}_{\vec{a}}\vec{c} = \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|} = \frac{17}{\sqrt{14}}$$

$\frac{17}{\sqrt{14}}$

(i) $\mathrm{proj}_{\mathbf{a}}(\mathbf{c})$ $= \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2}\vec{a}$

$$= \frac{17}{14}\langle 3,2,1\rangle$$

$\frac{17}{14}\langle 3,2,1\rangle$

5. (5 points) What is the equation of the plane perpendicular to $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and containing the point $P(2, -5, 1)$?

$\vec{n} \cdot (\vec{r} - \vec{p}) = 0$ i.e.

$2(x-2) - 3(y+5) + 5(z-1) = 0$

Equation is $2(x-2) - 3(y+5) + 5(z-1) = 0$ or $2x - 3y + 5z = 24$

6. (8 points) Find the equation of the plane through the points $P = (1, 1, 0)$, $B = (1, 0, 1)$ and $C = (0, 1, 1)$.

Equation is $x + y + z = 2$

To find a noral compute

$\vec{n} = \vec{PB} \times \vec{PC}$

$= \langle 0, -1, 1 \rangle \times \langle -1, 0, 1 \rangle$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix}$

$= \left\langle \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} \right\rangle = \langle -1, -1, -1 \rangle$

so equation is

$-1(x-1) - 1(y-1) - \cancel{1}1(z-0) = 0$

or $x + y + z = 2$

7. (7 points) Find the parametric form of the equation of the line through $\overset{P}{(1, 2, 3)}$ and $\overset{Q}{(6, 4, 1)}$.

$x = 1 + 5t,\ y = 2 + 2t,\ z = 3 - 2t$

The line through P and Q is

$\vec{r}(t) = \vec{p} + t\,\vec{PQ}$

$= \langle 1, 2, 3 \rangle + t \langle 5, 2, -2 \rangle$

In parametric form

$x = 1 + 5t$

$y = 2 + 2t$

$z = 3 - 2t$

There are other correct ans.

8. (5 points) What a the vector equation of the line of intersection of the planes $x + y = 2$ and $y + 2z = 6$?

$\vec{r}(t) = \langle 2 - 2t, 2t, 3 - t \rangle$

There are other correct answers

$x + y = 2$

$y + 2z = 6$

express x, z in terms of y

$x = 2 - y$

$z = 3 - \frac{1}{2} y$

Now let $y = 2t$ to get

$x = 2 - 2t$

$y = 2t$

$z = 3 - t$

In vector form this is

9. (8 points)

(a) For vectors $\mathbf{a}$ and $\mathbf{b}$ expand the expression $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$.

$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$

$= \vec{a}\cdot\vec{a} - \cancel{\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a}} - \vec{b}\cdot\vec{b}$

$= |\vec{a}|^2 - |\vec{b}|^2$

$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) =$ $|\vec{a}|^2 - |\vec{b}|^2$

(b) Use your answer to part (a) to show that if $\mathbf{a}$ and $\mathbf{b}$ have the same length, then $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are orthogonal. (Recall that two vectors are orthogonal if and only if their dot product is zero.)

We just need to show $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0$

By part (a)

$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = 0$ as $|\vec{a}| = |\vec{b}|$

10. (7 points) What it the angle between the planes $x - y = 1$ and $3y + 3z = 14$?

This is the same as the angle between the normals. $\vec{n_1} = \langle 1,-1,0\rangle$ $\vec{n_2} = \langle 0,3,3\rangle$

$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\theta$

$0 - 3 + 0 = \sqrt{2}\sqrt{18}\cos\theta$ $\cos\theta = \frac{-3}{\sqrt{2}\sqrt{18}} = \frac{-3}{\sqrt{2}\,3\sqrt{2}} = -\frac{1}{2}$

Angle is $\frac{2\pi}{3}$ rad $= 120^\circ$

11. (5 points) What is the distance between the points $P(1,2,3)$ and $Q(2,-3,5)$?

Distance $= \sqrt{(1-2)^2 + (2-(-3))^2 + (3-5)^2}$

$= \sqrt{1^2 + 5^2 + 2^2}$

$= \sqrt{1+25+4} = \sqrt{30}$

The distance is $\sqrt{30}$

12. (5 points) What is the symmetric form of the line through $P(3,-2,1)$ with direction vector $\mathbf{v} = \langle 4,2,-3\rangle$

$\frac{x-3}{3} = \frac{y+2}{2} = \frac{z-1}{3}$

The parametric form is

$x = 3 + 4t$

$y = -2 + 2t$

$z = 1 + 3t$

solve for t

$t = \frac{x-3}{3} = \frac{y+2}{2} = \frac{z-1}{3}$