Homework assigned Wednesday, February 16.

In class we discussed ***Taylor's Theorem*** which is that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has positive radius of convergence R, then f is analytic inside of its disk of convergence and the derivative is given by

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}.$$

Repeated application of this shows that f has derivatives of all orders and

$$f^{(k)}(z) = \sum_{n=0}^{\infty} n(n-1)\cdots(n-k+1) a_n (z - z_0)^{n-k}.$$

Moreover the coefficients a_n are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Problem 1. Here is some practice in using these results.

(a) Find the series expansion of $\cos(z)$ about $z = \pi/4$. *Hint:* Look at problem 6.24 on page 186 of the text.

(b) Find the expansion of e^{2z} about $z = i$.

(c) Find the expansion of $f(z) = \dfrac{1}{z-a}$ about $z = z_0$ where a and z_0 are complex numbers and $a \neq z_0$. *Hint:* This can be done either by using Taylor's theorem, or by doing a trick we have done before to reduce to geometric series

$$f(z) = \frac{1}{z-a} = \frac{1}{(z-z_0) - (a-z_0)} = \frac{-1}{a-z_0}\left(\frac{1}{1-\left(\frac{z-z_0}{a-z_0}\right)}\right).$$

As an example of Taylor's theorem we derived *Newton's Bionomial Theorem.* This that for any complex number α that

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$

where $\binom{\alpha}{0} = 1$ and for $n \geq 1$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Problem 2. This is for some practice with the bionomial theorem and just generally working with series.

(a) Find the expansion for $\dfrac{1}{(1+z)^2} = (1+z)^{-2}$.

(b) In your solution to part (a) replace z by z^3 to get a series for $\dfrac{1}{(1+z^3)^2}$.

(c) Find the expansion of for $\dfrac{1}{\sqrt{1+z}} = (1+z)^{-1/2}$.

(d) In your solution to the part (c) replace z by $-z^2$ to get a series for $\dfrac{1}{\sqrt{1-z^2}}$.

(e) Show that for any α that is not a nonnegative integer that

$$\lim_{n\to\infty} \frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} = 1.$$

(f) Use part (e) to find the radius of convergence of the series for $(1+z)^\alpha$.

(g) Use that $(1+z)(1+z)^\alpha = (1+z)^{\alpha+1}$ to show

$$\binom{\alpha}{n-1} + \binom{\alpha}{n} = \binom{\alpha+1}{n+1}$$

holds for $n \geq 1$. *Hint:* Start by

$$\begin{aligned}(1+z)(1+z)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n + z\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \\ &= \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n + \sum_{n=0}^{\infty} \binom{\alpha}{n} z^{n+1}\end{aligned}$$

do a change of index in the second sum and combine.

(h) (This one is a bit of a challenge problem.) Show

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} = \binom{\alpha+\beta}{n}$$

Hint: We know $(1+z)^\alpha(1+z)^\beta = (1+z)^{\alpha+\beta}$. Go back and look at the proof we did at the begining of the term to show $e^z e^w = e^{z+w}$. Conbining the ideas in that calculation with $(1+z)^\alpha(1+z)^\beta = (1+z)^{\alpha+\beta}$ should do the trick.

Finally use of series often helps with problems where we one might have been tempted to use l'Hôpital's rule. Here are few examples

$$\begin{aligned}\frac{\sin(2z)}{z} &= \frac{1}{z}\sin(2z) = \frac{1}{z}\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right) \\ &= 2 - \frac{2^3 z^2}{3!} + \frac{2^5 z^4}{5!} - \cdots\end{aligned}$$

and now it is clear that

$$\lim_{z\to 0} \frac{\sin(2z)}{z} = \lim_{z\to 0}\left(2 - \frac{2^3 z^2}{3!} + \frac{2^5 z^4}{5!} - \cdots\right) = 2.$$

Problem 3.

(a) Find the terms of the series for $\dfrac{e^z - 1 - z}{z^2}$ up to degree three. That is stop at the z^3 term.

(b) What is $\lim_{z\to 0} \dfrac{e^z - 1 - z}{z^2}$?

(c) Find the expansion for $\dfrac{\cos(2z) - 1 + 2z^2}{z^4}$ up to terms of degree 3.

(d) What is $\lim_{z\to 0} \dfrac{\cos(2z) - 1 + 2z^2}{z^4}$?