Homework assigned Monday, March 26.

The first problems here are working up to the proof of the fundamental theorem of algebra.

We know the ***triangle inequality*** for complex numbers

$$|z + w| \le |z| + |w|.$$

Problem 1. Use the triangle inequality to show for any complex numbers a, b that

$$|a + b| \ge |a| - |b|.$$

Hint: In the triangle inequality let $z = a + b$ and $w = -b$.

Problem 2. Use the last problem repeatedly to show

$$|a + b_1 + b_2 + \cdots b_n| \ge |a| - |b_1| - |b_2| - \cdots - |b_n|.$$

Instead of working with polynomials of degree n, it will simplify notation if we work with polynomials of degree 3. All the basic ideas are the same.

Problem 3. Let $p(z) = z^3 + b_2 z^2 + b_1 z + b_0$. Show

$$|p(z)| \ge |z|^3 \left(1 - \frac{|b_2|}{|z|} - \frac{|b_1|}{|z|^2} - \frac{|b_0|}{|z|^3}\right).$$

Problem 4. With notation as in Problem 3 show that if $R = \max\{1, 6|b_2|, 6|b_1|, 6|b_0|\}$ then show that for $|z| \ge R$ (that is $|z| \ge 1$, $|z| \ge 6|b_2|$, $|z| \ge 6|b_1|$) that the following hold

(a) $\dfrac{1}{|z|^3} \le \dfrac{1}{|z|^2} \le \dfrac{1}{|z|} \le 1$. *Hint:* This only uses $|z| \ge 1$.

(b) $\dfrac{|b_2|}{|z|} \le \dfrac{1}{6}$. *Hint:* This uses $|z| \ge 6|b_2|$.

(c) $\dfrac{|b_1|}{|z|^2} \le \dfrac{1}{6}$. *Hint:* This uses $|z| \ge 6|b_1|$ and part (a).

(d) $\dfrac{|b_0|}{|z|^3} \le \dfrac{1}{6}$. *Hint:* This uses $|z| \ge 6|b_0|$ and part (a).

(e) $|p(z)| \ge \dfrac{|z|^3}{2} \ge \dfrac{1}{2}$. *Hint:* This uses parts (b), (c), (d) and Problem 3.

Recall:

Theorem 1 (Louisville's Theorem)**.** *A bounded entire function is constant. (That is if $f(z)$ is function that is analytic on all of* $\mathbf{C}$ *and so that there is a constant M with $|f(z)| \le M$, then $f(z)$ is constant.)*

We will now use this to prove

Theorem 2 (Fundamental Theorem of Algebra)**.** *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$ be a complex polynomial of degree $n \ge 1$. Then $p(z)$ has at least one complex root. That is there is at least one complex number r with $p(r) = 0$.*

To start we note that by dividing by a_n we have that solving

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

is the same as solving

$$z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \cdots + \frac{a_1}{a_n} z + \frac{a_0}{a_n} = 0$$

so there is no loss of generality in assuming that the lead coefficient of $p(z)$ is one. That is $p(z)$ is of the form

$$p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0.$$

And, just to simplify notation, we assume that $n = 3$, so

$$p(z) = z^3 + b_2 z^2 + b_1 z + b_0.$$

Assume, towards a contradiction, that $p(z)$ has no roots. That is $p(z) \neq 0$ for all $z \in \mathbf{C}$. Define a new function $f(z)$ by

$$f(z) = \frac{1}{p(z)}.$$

Problem 5. Explain why $f(z)$ is an entire function.

Problem 6. Let R be as in Problem 4. Show

$$|z| \geq R \qquad \text{implies} \qquad |f(z)| \leq 2.$$

Problem 7. The function $|f(z)|$ is continuous on the closed bounded set $\{z : |z| \leq R\}$, so there is a constant C such that

$$|z| \leq R \qquad \text{implies} \qquad |f(z)| \leq C.$$

(This is a basic fact from Mathematics 554, so you don't have to prove it, just copy it down to get credit.)

Problem 8. Let R be as in Problem 4 and set $M = \max\{2, R\}$. Combine Problems 6 and 7 to show

$$|f(z)| \leq M$$

for all $z \in C$.

Problem 9. Now show that $f(z) = \dfrac{1}{p(z)}$ is constant and therefore $p(z)$ is also constant.

And finally

Problem 10. To finish the proof explain why the assumption $p(z)$ has no roots leads to a contradiction.