Homework assigned Wednesday, January 25.

Problem 1. Find the set where the following series converge and draw a picture of it.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$.

(b) The series $\cos(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{z^2}{3^n(n^2+1)}$.

Problem 2. We are now using for our official definitions of $\cos(z)$ and $\sin(z)$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

We also know that the exponential satisfies the basic identity

$$e^{z+w} = e^z e^w.$$

This identity and the definitions above let us prove about all the basic trigonometric identities in a straightforward manner. For example use the definition to do the following

(a) Simplify $\cos^2(z) + \sin^2(z)$ (as in the case of z being real, the answer is 1, but you need to show that it also holds for complex z.)

(b) Simplify $\cos(z)\cos(w) - \sin(z)\sin(w)$.

Problem 3. Show that if y is a real number, show that $\cos(iy)$ is a positive real number.

Problem 4. If $z = x + iy$ then

$$e^z = e^{x+iy} = e^x e^i y$$

and thus if $re^{i\theta}$ is the polar form of e^z, then $r = e^x$ and $\theta = y + 2n\pi$ for some integer n. Use this to find all solutions to the following

(a) $e^z = 1$.

(b) $e^z = -1$.

(c) $e^z = 1 - i$.

Recall that for a real number t that $\cosh(t)$ and $\sinh(t)$ are defined by

$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

Let $z = x + iy$. Then we have

$$\begin{aligned}
\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\
&= \frac{e^{-y+ix} + e^{y-ix}}{2} \\
&= \frac{e^{-y}e^{ix}}{2} + \frac{e^{y}e^{-ix}}{2} \\
&= \frac{e^{-y}\left(\cos(x) + i\sin(x)\right)}{2} + \frac{e^{y}\left(\cos(x) - i\sin(x)\right)}{2} \\
&= \cos(x)\left(\frac{e^{-y} + e^{y}}{2}\right) + i\sin(x)\left(\frac{e^{-y} - e^{y}}{2}\right) \\
&= \cos(x)\cosh(y) - i\sin(x)\sinh(y).
\end{aligned}$$

Therefore the real part of $\cos(z)$ is $\cos(x)\cosh(y)$ and the imaginary part is $-\sin(x)\sinh(y)$.

Problem 5. Do a similar calculation to find the real and imaginary parts of $\sin(z)$.