Homework assigned Friday, February 3.

Here we will mostly be looking at consequences of the Cauchy-Riemann equations. That is if $f = u + iv$ in an open set, then

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x.$$

Definition 1. Let U be an open set in the complex plane $\mathbf{C}$. Then a function $h\colon U \to \mathbf{R}$ is ***harmonic*** iff

$$h_{xx} + h_{yy} = 0.$$

(It is being assumed that the first and second partial derivatives of h exist and are continuous.) $\square$

A very important result is that the real and imaginary parts of an analytic function are harmonic. To be precise

Theorem 2. *Let $f = u + iv$ be analytic in the open set U. Assume that u and v have continuous first and second partial derivatives. Then both u and v are harmonic.*

While this is important it is not hard:

Problem 1. Prove the last theorem. *Hint:* It is a more or less direct consequence of the Cauchy-Riemann equations. As a start note

$$u_{xx} = (u_x)_x = (v_y)_x = v_{xy}$$

with a similar formula for u_{yy} in terms of v_{xy}. If you want more or a hint see page 88 of the text. $\square$

Here is a variant of some of the problems we did in class today.

Problem 2. Let $f = u + iv$ be analytic in a connected open set U. Assume that $u^2 - v^2 = c$ where $c \neq 0$ is a constant. Show f is constant. *Hint:* It is enough to show that u and v are constant. And for that it is enough to show $u_x = u_y = 0$ and $v_x = v_y = 0$. Take the first partial derivatives of the equation $u^2 - v^2 = c$ with respect to x and y and use the Cauchy-Riemann equations. $\square$

A consequence of our proof of the Cauchy-Riemann equations is

Proposition 3. *Let $f = u + iv$ be analytic in an open set U. Then the derivative of f is give by either of the formulas*

$$f' = u_x + iv_x \qquad \textit{and} \qquad f' = v_y - iu_y$$

(In practice we usually just use $f' = u_x + iv_x$.) $\square$

Here is an example similar to an example we did in class, if $f(z) = e^{2z}$, then

$$f(z) = e^{2x}\cos(2y) + ie^{2x}\sin(2y) = u + iv.$$

Thus

$$f'(z) = u_x + iv_x = 2e^{2x}\cos(2y) + i2e^{2x}\sin(2y) = 2e^{2z}$$

just as we expected.

Problem 3. Use Proposition 3 to show the following (which we are all familiar with for real values, but which we still need to verify for complex values.)

(a) If $f(z) = \cos(z)$, then $f'(z) = \sin(z)$.
(b) If $f(z) = \sin(z)$, then $f'(z) = \cos(z)$.
(c) If $f(z) = \log(z)$, then $f'(z) = \dfrac{1}{z}$.

While at this point is not clear there is much relationship between analytic function and functions that can be expressed as a convergent power series, it will turn out that the two are closely related. Here is a start on that

Proposition 4. *Consider the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Assume this converges for $z = z_1$. Then the series converges for all z with $|z| < |z_1|$.

Problem 4. Prove this. Before starting we make a few observations. As the series for $f(z_1)$ converges, the terms go to zero. That is $\lim_{n\to\infty} a_n z_1^n = 0$. This implies the terms are bounded, that is there is a constant, C, so that

$$|a_n z_1^n| \le C.$$

Define

$$r = \frac{|z|}{|z_1|} = \left|\frac{z}{z_1}\right|.$$

By hypothesis $|z| < |z_1|$, so

$$r < 1.$$

Thus by our basic results about geometric series

$$\sum_{n=0}^{\infty} C r^n < \infty \tag{1}$$

Now proceed with the proof as follows.

(a) Show $|a_n z^n| \le C r^n$. *Hint:* $|a_n z^n| = |a_n z_1^n| |z/z_1|^n$.
(b) Finish the proof by use of the comparison theorem (look this up if you have forgotten it) part (a) and (1).

Problem 5 (Not to be handed in). Review the definition of the gradient and the chain rule for functions of two variables. In particular that the gradient of a function is orthogonal to the level curves of the function.