Homework assigned Friday, April 6.

Let be the annulus $D = \{z : r < |z - z_0| < R\}$. Then in class today we proved

Theorem 1 (Laurent expansion)**.** *Let $f(z)$ be analytic in D. Then $f(z)$ has a convergent* ***Laurent expansion***

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

which holds for all $z \in D$. □

We also proved

Lemma 2. *Let k be an integer. Then for any r_0*

$$\int_{|z-z_0|=r_0} (z - z_0)^k \, dz = \begin{cases} 2\pi i, & k = -1; \\ 0, & k \neq -1. \end{cases}$$

□

We used this to do the following calculation, where $f(z)$ is as in Theorem 1 and $r < r_0 < R$

$$\begin{aligned}
\int_{|z-z_0|=r_0} f(z)\, dz &= \int_{|z-z_0|=r_0} \left(\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \right) dz && \text{(Replace } f(z) \text{ by its series)} \\
&= \sum_{n=-\infty}^{\infty} a_n \int_{|z-z_0|=r_0} (z - z_0)^n \, dz && \text{(Integrate term by term)} \\
&= a_{-1} \int_{|z-z_0|=r_0} (z - z_0)^{-1} \, dz && \text{(Terms with } n \neq -1 \text{ are zero)} \\
&= 2\pi i a_{-1}. && \left(\text{As } \int_{|z-z_0|=r_0} (z - z_0)^{-1} \, dz = 2\pi i\right)
\end{aligned}$$

Theorem 3. *Let $f(z)$ be analytic in the annulus defined above and have the Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Then the coefficients are given by

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=r_0} \frac{f(z)\, dz}{(z - z_0)^{k+1}}$$

Problem 1. Prove this. *Hint:* Take the expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

and multiply divide by $(z - z_0)^{k+1}$ to get

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^{n-k-1}.$$

Now proceed as in the calculation above. □

Problem 2. Know the statement of the Laurent expansion as there may be a quiz on it on Monday.