Homework assigned Monday, April 2.

First some review.

Proposition 1. *Let $f = u + iv$ be analytic in a connected domain D. Assume that $|f(z)|$ is constant. Then $f(z)$ is constant.*

Problem 1. Prove this along the following lines.

(a) If $|f(z)|$ is constant then show $u^2 + v^2 = c$ for some real constant c.
(b) If $c = 0$ show $f(z)$ is the constant function 0.
(c) If $c \neq 0$ use the Cauchy-Riemann equations to show $f(z)$ is constant.

The following is a special case of something we proved in class last week.

Proposition 2. *Let $f(z)$ be continuous on the circle $|z - z_0| = r$. Then*

$$\left| \int_0^{2\pi} f(z_0 + re^{it})\, dt \right| \leq \int_0^{2\pi} |f(z_0 + re^{it})|\, dt$$

and if equality holds, then $|f(z_0 + re^{it})|$ is constant (as a function of t.)

Theorem 3 (Maximum modulus principle)**.** *Let $f(z)$ be analytic on the closure of $D(z_0, R)$ and assume that $|f(z)|$ has a maximum at $z = z_0$ (that is $|f(z)| \leq |f(z_0)|$ for $z \in D(z_0, R)$). Then $f(z)$ is constant in $D(z_0, R)$.*

Problem 2. Prove this along the following lines.

(a) If $0 < r < R$ use the mean value property of analytic functions to write

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

(You don't have to prove this). Then use the argument we gave in class to show

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|\, dt \leq |f(z_0)|.$$

(b) Explain why for equality to hold in the second of these inequalities we have $|f(z_0 + re^{it})| = |f(z_0)|$ for $0 \leq t \leq 2\pi$.
(c) By varying $r \in (0, R)$ and $t \in [0, 2\pi]$ in part (b) show that $|f(z)| = |f(z_0)|$ for $z \in D(z_0, R)$.
(d) Now use Proposition 1 to show $f(z)$ is constant in $D(z_0, R)$.

Here is anther form of the maximum modulus principle.

Problem 3. Let D be a bounded domain and let $f(z)$ be analytic on $\overline{D}$ (the closure of D.) Then $f(z)$ achieves its maximum on $\overline{D}$ on the boundary, ∂D, of D.

Problem 4. Prove this along the following lines.

(a) If $|f(z)|$ is constant, then $f(z)$ is constant and so the maximum of $|f(z)|$ occurs at all points of ∂D. In particular it occurs on the boundary.

(b) So assume that $f(z)$ is not constant. Assume, toward a contradiction that the maximum of $|f(z_0)|$ occurs in D rather than on ∂D. Then get a contradiction by showing that $f(z)$ is constant.

Proposition 4 (Minimum modulus principle)**.** *Let $f(z)$ be analytic on the closure of $D(z_0, R)$ and assume that $|f(z)|$ has a minimum at $z = z_0$ (that is $|f(z)| \geq |f(z_0)|$ for $z \in D(z_0, R)$). Then either $f(z)$ is constant or $f(z_0) = 0$.*

Problem 5. Prove this. *Hint:* If $f(z_0) \neq 0$ then show $f(z) \neq 0$ for all $z \in D(z_0, R)$ and then apply the maximum modulus principle to $g(z) = 1/f(z)$.