Homework for Spring Break.

These problems will be due on the Wednesday after Spring. So you can ask question about them on the Monday when we return. The assignment is not so long that if you work on it a three or four times over the break that it will be a big time sink, but it is long enough that it will be hard to do between Monday and Wednesday after the break. So it good to work on it over the break.

There will be a quiz on Monday after the break. It will cover the following.

(1) Knowing the statement of Cauchy's Theorem.
(2) Knowing the statement of Green's Theorem.
(3) Knowing how to use Green's together with the Cauchy-Riemann equations to prove Cauchy's Theorem.
(4) Knowing the statement of the Cauchy Integral Theorem.

Here is a summary of part of the main plot, at least as related to analytic functions, to date.

Definition 1. A complex valued function $f(z)$ is ***analytic*** on an open subset D of $\mathbf{C}$ iff it is complex differentiable in D. That is for all $z \in D$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

exists. □

By computing the limit in the definition of $f'(z)$ in two ways, first by letting $\Delta z = \Delta x \to 0$ through real values, and second by letting $\Delta z = i\Delta y \to 0$ go to zero through imaginary values we derived

Theorem 1 (Cauchy-Riemannian equations)**.** *If a function $f(z) = u + iv$ has continuous first partial derivatives in the open set D, then $f(z)$ is analytic if and only if u and v satisfy*

$$u_x = v_y, \qquad u_y = -v_x.$$

□

Then, from our vector calculus class, we recalled

Theorem 2 (Green's Theorem)**.** *Let D be a bounded domain in $\mathbf{C}$ with a nice boundary ∂D. Then if $P(x,y)$ and $Q(x,y)$ are functions on the closure of D that have continuous partial derivatives then*

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (-P_y + Q_x)\;dx\,dy.$$

□

Green's theorem and the Cauchy-Riemann equations then combine in an easy and natural way to give:

Theorem 3 (Cauchy's Theorem)**.** *Let D be a bounded domain with nice boundary and $f(z)$ a function that is analytic on the closure of D. Then*

$$\int_{\partial D} f(z)\,dz = 0.$$

□

While I have mentioned the following terms in class I may have been a bit vague, so here are formal definitions.

Definition 2. A ***domain*** in **C** is a connected open set D. □

Definition 3. A domain is ***simply connected*** iff it has no holes in it. (Figure 1 shows some simply connected domains and Figure 2 shows some non-simply connected domains.) □

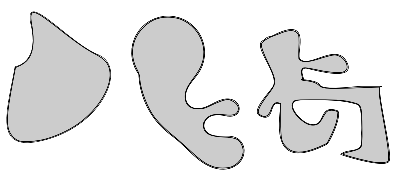

FIGURE 1. Three simply connected domains.

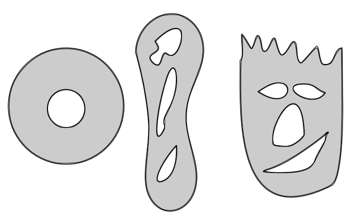

FIGURE 2. Three non-simply connected domains.

We used Cauchy's Theorem to show

Theorem 4. *Let $f(z)$ be analytic in a simply connected domain D. Then $f(z)$ has an antiderivative in D. That is there is a function $F(z)$ defined in D with $F'(z) = f(z)$ in D.* □

Remark 1. We only showed this in detail when the domain was starlike. But it is not hard to extend the result to general simply connected domains.

Definition 4. Let $f(z)$ be analytic in the domain D. Then $g(z)$ is a ***logarithm*** of $f(z)$ in D iff $e^{g(z)} = f(z)$. □

Problem 1. Explain why this is the proper definition of $g(z)$ being a logarithm of $f(z)$. □

Problem 2. Show that if $g(z)$ is a logarithm of $f(z)$ in D then for any integer n the function $h(z) = g(z) + 2\pi n i$ is also a logarithm of $f(z)$. Thus logarithms are never unique. $\square$

There is an easy condition that implies that $f(z)$ has a logarithm.

Theorem 5. *Let D be a simply connected domain and $f(z)$ a function that is analytic in D and nonvanishing in D (that is $f(z) \neq 0$ for all $z \in D$). Then $f(z)$ has a logarithm in D.*

Problem 3. Prove this along the following lines. (This proof is motivated by noting that if $g(z) = \log f(z)$ then we should have $g'(z) = f'(z)/f(z)$.)

(a) While we have not shown it yet[1], it is true that if $f(z)$ is analytic, so is its derivative $f'(z)$. Explain why $\dfrac{f'(z)}{f(z)}$ is analytic in D. *Hint:* This really does not involve any more than saying $f(z) \neq 0$ in D.

(b) Explain why $\dfrac{f'(z)}{f(z)}$ has an anti-derivative. Call this anti-derivative $g_1(z)$. *Hint:* Theorem 4.

(c) Show that $f(z)e^{-g_1(z)}$ is constant. *Hint:* About the most natural way to show that a function is constant is to show its derivative is zero. Note that

$$\frac{d}{dz}\left(f(z)e^{-g_1(z)}\right) = f'(z)e^{-g_1(z)} - f(z)g_1'(z)e^{-g_1(z)}.$$

and, as $g_1(z)$ is an anti-derivative of $f'(z)/f(z)$

$$g_1'(z) = \frac{f'(z)}{f(z)}.$$

(d) From part (c) we have $f(z)e^{-g_1(z)} = c$ for some non-zero complex constant c. Thus $f(z) = ce^{g_1(z)}$. Show that there is a complex constant a so that $g(z) = g_1(z) + a$ is a logarithm of $f(z)$. $\square$

We can also take roots in simply connected domains.

Theorem 6. *Let $f(z)$ be analytic and nonvanishing in the simply connected domain D and let n be a positive integer. Then there is an analytic function $h(z)$ with*

$$h(z)^n = f(z).$$

(Thus when $n = 2$, $h(z)$ is a square root of $f(z)$, when $n = 3$, $h(z)$ is a cube root of $f(z)$ etc.)

Problem 4. Prove this. *Hint:* Let $g(z)$ be a logarithm of $f(z)$ and consider the function $h(z) = e^{g(z)/n}$. $\square$

We also used Cauchy's Theorem to prove

[1] But you will in Problem 6 below.

Theorem 7 (Cauchy Integral Fromula)**.** *Let D be a bounded domain with nice boundary and $f(z)$ analytic on the closure of D. Then for any $a \in D$*

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-a}\, dz.$$

□

We now derive an analogous formula for the derivative of $f(z)$.

Problem 5. Let D be a bounded domain with nice boundary and let $f(z)$ be analytic in the closure of D.

(a) For any $a, h \in \mathbf{C}$ show

$$\frac{1}{z-(a+h)} - \frac{1}{z-a} = \frac{h}{(z-(a+h))(z-a)}.$$

(b) If $a \in D$ and h is so small that $a+h$ is also in D show

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-(a+h))(z-a)}\, dz.$$

(c) In part (b) take the limit at $h \to 0$ to show

$$f'(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-a)^2}\, dz.$$

□

Now for the second derivative.

Problem 6. Let D be a bounded domain with nice boundary and let $f(z)$ be analytic in the closure of D.

(a) For any $a, h \in \mathbf{C}$ show

$$\frac{1}{(z-(a+h))^2} - \frac{1}{(z-a)^2} = \frac{h\big(2(z-a)-h\big)}{(z-(a+h))^2(z-a)^2}.$$

(b) If $a \in D$ and h is so small that $a+h$ is also in D show

$$\frac{f'(a+h)-f'(a)}{h} = \frac{1}{2\pi i} \int_{\partial D} \frac{\big(2(z-a)-h\big)f(z)}{(z-(a+h))^2(z-a)^2}\, dz.$$

(c) In part (b) take the limit as $h \to 0$ to show that $f''(a)$ exists and give a formula for $f''(a)$. □

As a was any point of d this shows that $(f')' = f''$ exists at all points of D. That is f' is also analytic in D. So we have

Theorem 8. *Let $f(z)$ be analytic in a domain D. Then the derivative, f', of f is also analytic in D.* □