We give another basic result about power series.

Theorem 1 (Abel's Continuity Theorem 1827). *Let*

$$\sum_{k=1}^{\infty} a_k$$

be a convergent series of real numbers. (It is not assumed that it is absolutely convergent.) Then the series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

converges for $|x| < 1$ and the one sides limit as $x \to 1^-$ is what it should be:

$$\lim_{x\to 1^-} f(x) = \sum_{k=0}^{\infty} a_k$$

Problem 1. Show that this series converges for $|x| < 1$. Hint: As $\sum_{k=1}^{\infty} a_k$ converges, we have $\lim_{k\to\infty} a_k = 0$ and therefore there is a constant M so that $|a_k| \le M$ for all k. Now compare the series for $f(x)$ to the geometric series $\sum_{k=0}^{\infty} M|x|^k$ which converges for $|x| < 1$. □

Some more notation, for $n = 0, 1, 2, \ldots$ set

$$A_n := \sum_{k=0}^{n} a_k, \qquad A = \sum_{k=0}^{\infty} a_k$$

We also set

$$A_{-1} = 0.$$

Problem 2. Show that with these definitions

$$a_k = A_k - A_{k-1}$$

for $k = 0, 1, 2, \ldots$. □

The statement of the Abel's theorem can now be rewritten as saying that for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$1 - \delta < x < 1 \implies |f(x) - A| < \varepsilon. \tag{1}$$

The following clever tricp is due to Abel.

Problem 3. Show that for $|x|$ that

$$f(x) = (1 - x) \sum_{k=0}^{\infty} A_k x^k.$$

Hint: Justify the following calculation

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty}(A_k - A_{k-1})x^k \\ &= \sum_{k=0}^{\infty} A_k x^k - \sum_{k=0}^{\infty} A_{k-1}x^k \\ &= \sum_{k=0}^{\infty} A_k x^k - \sum_{k=-1}^{\infty} A_k x^{k+1} \\ &= \sum_{k=0}^{\infty} A_k(x^k - x^{k+1}) \\ &= (1-x)\sum_{k=0}^{\infty} A_k x^k. \end{aligned}$$

□

The next trick uses that we know how to sum geometric series.

Problem 4. Show that

$$A = (1-x)\sum_{k=0}^{\infty} A x^k.$$

□

Putting these formulas together we have

$$f(x) - A = \sum_{k=0}^{\infty}(A_k - A)x^k.$$

Let N be a positive integer to be chosen Shortly. Then we split the sum for $f(x) - A$ into two pieces

$$f(x) - A = (1-x)\sum_{k=0}^{N}(A_k - A)x^k + (1-x)\sum_{k=N+1}^{\infty}(A_k - A)x^k.$$

As $\lim_{k\to\infty} A_k = A$, we can choose N so that

$$k \ge N \quad \Longrightarrow \quad |A_k - A| < \frac{\varepsilon}{2}.$$

Problem 5. Show that for this N and $0 \le x < 1$

$$\left|(1-x)\sum_{k=N+1}^{\infty}(A_k - A)x^k\right| < \frac{\varepsilon}{2}.$$

Hint: Here is a start:

$$\begin{aligned} \left|(1-x)\sum_{k=N+1}^{\infty}(A_k - A)x^k\right| &\le (1-x)\sum_{k=N+1}^{\infty}|A_k - A|x^k \\ &\le (1-x)\sum_{k=N+1}^{\infty}\left(\frac{\varepsilon}{2}\right)x^k \end{aligned}$$

and you can now sum a geometric series and use that $0 \le x < 1$. □

Problem 6. If $0 \le x < 1$ use the last problem and a formula above to show

$$|f(x) - A| \le (1-x)\sum_{k=0}^{N} |A_k - A| x^k + \frac{\varepsilon}{2} \le (1-x)\sum_{k=0}^{N} |A_k - A| + \frac{\varepsilon}{2}$$ □

Problem 7. Complete the proof of Abel's Theorem. Hint: Let $M := \sum_{k=0}^{N} |A_k - A|$ and set $\delta = \varepsilon/(2M)$ and verify that with this δ the implication (1) holds. □

Problem 8. Recall that we have seen that

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

for $|x| < 1$. But we would like to let $x = 1$ and conclude that

$$\ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \tag{2}$$

$$\frac{\pi}{4} = \arctan(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \tag{3}$$

Use Abel's continuity theorem to show that series expansions (2) and (3) for ln(2) and $\pi/4$ hold. (You may assume that ln is continuous at $x = 2$ and that arctan is continuous at $x = 1$.) *Hint:* You don't have to say much more than the theorem clearly applies. □