Mathematics 554 Test #1

Name: **Answer Key**.

1. State the following:

(a) The definition of $\lim_{n\to\infty} a_n = A$.

Solution: For all $\varepsilon > 0$ there is a N such that

$$n > N \quad \implies \quad |a_n - A| < \varepsilon.$$

(b) The definition of $\langle a_k \rangle_{k=1}^{\infty}$ being a ***Cauchy sequence***.

Solution: For all $\varepsilon > 0$ there is a N such that

$$m, n > N \quad \implies \quad |a_m - a_n| < \varepsilon.$$

(c) The definition of $\limsup_{n\to\infty} a_n$.

Solution:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

(d) If $\phi = \sum_{j=1}^{n} a_j \chi_{I_j}$ is a step function, define the ***integral*** $\int_a^b \phi(x)\,dx$ of ϕ.

Solution:

$$\int_a^b \phi(x)\,dx = \sum_{j=1}^{n} a_j |I_j|$$

where $|I_j|$ is the length of the interval I_j.

(e) If f is a bounded function on the interval $[a,b]$ define the ***upper integral*** $\overline{\int_a^b} f(x)\,dx$.

Solution:

$$\overline{\int_a^b} f(x)\,dx = \inf\left\{ \int_a^b \phi(x)\,dx : \phi \text{ is a step function and } f \le \phi \text{ on } [a,b] \right\}$$

(f) State both forms of the ***Fundamental Theorem of Calculus***.

Solution: **First form:** If f is integrable on $[a,b]$ and F is defined on $[a,b]$ by

$$F(x) = \int_a^x f(t)\,dt$$

when at any points $x \in (a, b)$ where f is continuous the function F is differentable at x and

$$F'(x) = f(x).$$

Second form: If f is continuous on the closed bounded interval $[a, b]$ and F is a function such that F is continuous on $[a, b]$ and $F'(x) = f(x)$ on (a, b) then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

2. Define a function F on $\mathbb{R}$ by

$$F(x) = \int_{-1}^{x^2} \cos(t^3)\, dt.$$

Find $F'(x)$.

Solution: Let $G(x) = \displaystyle\int_{-1}^{x} \cos(t^3)\, dt$. Then by the Fundamental Theorem of Calculus $G'(x) = \cos(x^3)$. But $F(x) = G(x^2)$ and so by the chain rule

$$F'(x) = G'(x^2)(x^2)' = \cos((x^2)^3)(2x) = 2x\cos(x^6).$$

3. Explain briefly (just a few sentences that quote the appropriate theorem) why the function $f(x) = \sin(x^3)$ is integrable on the interval $[-1, 2]$.

Solution: We know that every continuous function on a closed bounded interval is integrable on that interval. The function $f(x) = \sin(x^3)$ is continuous on the closed bounded interval $[-1, 2]$ and therefore $\int_{-1}^{2} \cos(x^3)\, dx$ exists.

4. Find the following limits. You do not have to prove your answers.

(a) $\lim_{n\to\infty} n^{2/3}\left(\sqrt[3]{n+4}-\sqrt[3]{n}\right)$

Solution:

$$\begin{aligned}
&\lim_{n\to\infty} n^{2/3}\left(\sqrt[3]{n+4}-\sqrt[3]{n}\right) \\
&\quad= \lim_{n\to\infty} n^{2/3}\left(f(n+4)-f(n)\right) && \text{(Where } f(x)=x^{1/3}\text{)} \\
&\quad= \lim_{n\to\infty} n^{2/3} f'(n+\xi)((n+4)-4) && \text{(With } 0<\xi<4 \text{ by MVT.)} \\
&\quad= \lim_{n\to\infty} n^{2/3}\frac{1}{3}(n+\xi)^{-2/3}(4) && \text{(As } f'(x)=(1/3)x^{-2/3}\text{)} \\
&\quad= \frac{4}{3}\lim_{n\to\infty}\left(\frac{n}{n+\xi}\right)^{2/3} \\
&\quad= \frac{4}{3}\lim_{n\to\infty}\left(\frac{1}{1+(\xi/n)}\right)^{2/3} \\
&\quad= \frac{4}{3}\left(\frac{1}{1+0}\right)^{2/3} && \text{(As } 0<\xi/n<4/n\text{)} \\
&\quad= \frac{4}{3}.
\end{aligned}$$

(b) $\lim_{n\to\infty}\frac{1}{n^3}\sum_{j=1}^{2n} j^2.$

Solution: Write

$$\frac{1}{n^3}\sum_{j=1}^{2n} j^2 = \sum_{j=1}^{2n}\left(\frac{j}{n}\right)^2\frac{1}{n}.$$

Then this is a Riemann sum for the function $f(x)=x^2$ on the interval $[0,2]$ when it is divided into $2n$ pieces (so that $\Delta x=(2-0)/(2n)=1/n$). Thus

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{j=1}^{2n} j^2 = \int_0^2 x^2\,dx = \frac{8}{3}.$$

5. (a) State the ***Bolzano-Weierstrass Theorem***.

Solution: A bounded sequence of real numbers has a convergent subsequence.

(b) If f is a continuous function on $[a,b]$ prove that f achieves its maximum on $[a,b]$.

Solution: We know that continuous functions on closed bounded intervals are bounded. Therefore

$$M = \sup\{f(x) : x \in [a,b]\}$$

is finite. We wish to show that there is a $x_{\max} \in [a,b]$ with $f(x_{\max}) = M$. From the definition of M as a least upper bound, for each positive integer n there is an $a_n \in [a,b]$ such that

$$M - \frac{1}{n} < f(a_n) \le M.$$

As $[a,b]$ is bounded the sequence $\langle a_n \rangle_{n=1}^{\infty}$ is bounded and therefore by the Bolzano-Weierstrass Theorem it has convergent subsequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$. Let

$$x_{\max} = \lim_{k\to\infty} a_{n_k}.$$

Then $x_{\max} \in [a,b]$ as $[a,b]$ is closed. Also, as f is continuous,

$$f(x_{\max}) = \lim_{k\to\infty} f(a_{n_k}).$$

But we also have

$$M - \frac{1}{n_k} < f(a_{n_k}) \le M$$

and so by the squeeze lemma

$$f(x_{\max}) = \lim_{k\to\infty} f(a_{n_k}) = M$$

as required.

6. Let f continuous on $[0,2]$. Define

$$F(x) = \int_0^x f(t)\,dt.$$

Prove directly that $F'(1) = f(1)$.

Solution: We wish to show

$$F'(1) = \lim_{h\to 0} \frac{F(1+h) - F(1)}{h} = f(1).$$

Let $\varepsilon > 0$. As f is continuous at 1 there is a $\delta > 0$ such that

$$|t-1| < \delta \quad \Longrightarrow \quad |f(t) - f(1)| < \varepsilon.$$

If $|h| < \delta$

$$\begin{aligned}
\left| \frac{F(1+h) - F(1)}{h} - f(1) \right| &= \left| \frac{1}{h} \int_1^{1+h} f(t)\,dt - f(1) \right| \\
&= \left| \frac{1}{h} \int_1^{1+h} f(t)\,dt - \frac{1}{h} \int_1^{1+h} f(1)\,dt \right| \\
&= \left| \frac{1}{h} \int_1^{1+h} (f(t) - f(1))\,dt \right| \\
&\le \left| \frac{1}{h} \int_1^{1+h} |f(t) - f(1)|\,dt \right| \\
&< \left| \frac{1}{h} \int_1^{1+h} \varepsilon\,dt \right| \qquad \text{(as } |t-1| < \delta \text{ because } |h| < \delta.) \\
&= \varepsilon
\end{aligned}$$

Thus we have shown

$$|h| < 0 \quad \implies \quad \left| \frac{F(1+h) - F(1)}{h} - f(1) \right| < \varepsilon.$$

That is

$$F'(1) = \lim_{h \to 0} \frac{F(1+h) - F(1)}{h} = f(1).$$

7. Let f be continuous on all of $\mathbb{R}$ and let $\langle a_n \rangle_{n=1}^{\infty}$ be a sequence with $\lim_{n\to\infty} a_n = 0$. Prove directly that $\lim_{n\to\infty} f(a_n) = f(0)$.

Solution: Let $\varepsilon > 0$. As f is continuous at $x = 0$ there is $\delta > 0$ such that

$$|x - 0| < \delta \quad \implies \quad |f(x) - f(0)| < \varepsilon.$$

As $\lim_{n\to\infty} a_n = 0$ there is a N such that

$$n > N \quad \implies \quad |a_n - 0| < \delta.$$

Thus it $n > N$ we have $|a_n - 0| < \delta$ and so also $|f(a_n) - f(0)| < \varepsilon$. That is

$$n > N \quad \implies \quad |f(a_n) - f(0)| < \varepsilon$$

which is exactly the definition of $\lim_{n\to\infty} f(a_n) = f(0)$.