Mathematics 555

Test #2 Name: Answer Key.

1. (a) State the definition of what it means for a sequence of functions $\langle f_n\rangle_{n=1}^{\infty}$ from a set E to $\mathbf{R}$ to converge ***pointwise*** to $f\colon E\to\mathbf{R}$.

Solution: For each $x\in E$ we have $\lim_{n\to\infty} f_n(x)=f(x)$. □

Alternate Solution: $\forall x\in E\ \forall\varepsilon>0\ \exists N\,(n\ge N \implies |f_n(x)-f(x)|<\varepsilon)$
(Thus $N=N(x,\varepsilon)$ depends on both x and ε.) □

(b) State the definition of what it means for a sequence of functions $\langle f_n\rangle_{n=1}^{\infty}$ from a set E to $\mathbf{R}$ to converge ***uniformly*** to $f\colon E\to\mathbf{R}$.

Solution: $\forall\varepsilon>0\ \exists N\ \forall x\in E\,(n\ge N \implies |f_n(x)-f(x)|<\varepsilon)$.
(In this case $N=N(\varepsilon)$ only depends on ε, and not x.) □

(c) Give an example of a sequence of functions $f_n\colon[0,1]\to\mathbf{R}$ that converges pointwise, but not uniformly.

Solution: We gave two examples in class. The first was

$$f_n(x)=x^n$$

and the second was

$$f_n(x)=\frac{nx}{1+n^2x^2}.$$

See your notes for details. □

2. Which of the following converge. Explain what tests you used to determine convergence or divergence.

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{k+1}{k^2+1}$.

Solution: This series **diverges**. There were several ways to do it. Maybe the easiest is to let the given series be $\sum a_k$ and let $\sum b_k$ be the series $\sum 1/k$. Then

$$\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{k(k+1)}{k^2+1}=1.$$

So by the *limit comparison test* the two series either both converge or diverge. As $\sum_k b_k$ is harmonic series, it diverges. So $\sum_{k=1}^{\infty}\frac{k+1}{k^2+1}$ diverges. □

Remark: Any time you have a seris $\sum a_k$ were $a_k=\frac{P(k)}{Q(k)}$ where $P(x)$ and $Q(x)$ are polynomials doing a limit comparison with $1/k^p$ where $p=\deg Q(x)=\deg P(x)$ will work. □

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.

This **converges**. Having all these factorials suggests that the *ratio test* would be a good thing to try. If the series is $\sum a_n$ then

$$\text{ratio} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2n+2)!}\frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1.$$

As the ratio is less than 1, the series converges. □

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(k+5)}$.

Solution: As the numbers $a_k = \frac{1}{\ln(k+5)}$ decrease monotonically to zero the series $\sum_{k=1}^{\infty}(-1)^k a_k$ **converges** by the *alternating series test*. □

3. Find the radius of convergence of the following power series.

(a) $\displaystyle\sum_{k=0}^{\infty} 2^k \left(\frac{k+1}{k^2+2}\right)^k x^k$.

Solution: Let us try the root test.

$$\text{root} = \lim_{k\to\infty} \left| 2^k \left(\frac{k+1}{k^2+2}\right)^k x^k \right|^{\frac{1}{k}} = \lim_{k\to\infty} 2\left(\frac{k+1}{k^2+2}\right)|x| = 0.$$

This holds for all x and therefore the series converges for all x. Thus the radius of convergence is $r = \infty$. □

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{4^n} x^{2n}$.

Solution: Again let us try the root test

$$\text{root} = \lim_{n\to\infty} \left| \frac{n+1}{4^n} x^{2n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(n+1)^{1/n} x^2}{4} = \frac{x^2}{4}$$

as $\lim_{n\to\infty}(n+1)^{1/n} = \lim_{n\to\infty} e^{(1/n)\ln(n+1)} = e^0 = 1$. Thus the seris converges if $x^2/4 < 1$ and diverges if $x^2/4 > 1$. Thus the radius of convergence is $r = 2$. □

Alternate Solution: This time let us use the ratio test.

$$\text{ratio} = \lim_{n\to\infty} \left(\frac{(n+2)x^{2n+2}}{4^{n+1}}\right)\left(\frac{4^n}{(n+1)x^{2n}}\right) = \lim_{n\to\infty} \frac{(n+2)x^2}{4(n+1)} = \frac{x^2}{4},$$

and the rest is just as in the last solution. □

4. (a) Define what it means for $\sum_{n=1}^{\infty} a_n$ to be ***absolutely convergent***.

Solution: That the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges. □

(b) Define what it means for $\sum_{n=1}^{\infty} a_n$ to be ***conditionally convergent***.

Solution: The series $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges. □

Remark: As the series $\sum_{n=1}^{\infty} a_n$ can have terms of both signs I took off a point or two if you wrote "$\sum_{n=1}^{\infty} a_n < \infty$" rather than "$\sum_{n=1}^{\infty} a_n$ converges".

(c) Give an example of a series that is conditionally, but not absolutely convergent.

Solution: For any p with $0 < 1 \leq 1$ the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

is an example. It converges by the alternating series test, but the series of absolute values, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, diverges as it is p-seris with $p \geq 1$. □

5. The function f defined by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

has the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \cdots$$

and this series has infinite radius of convergence. (You can assume this without proof.)

(a) Does the 7-th derivative, $f^{(7)}(x)$, exist? Give a brief explanation. (This should only involve quoting a theorem or fact we have used.)

Solution: We have shown that for a power series $f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ that has radius of convergence r, then $f'(x)$ exists and also has a power series with the same radis of convergence. Applying this to $f(x)$ seven times gives that $f^{(7)}(x)$ exists for all $x \in \mathbf{R}$. □

(b) Give the first five terms of the power series for $f'(x)$ and quote a theorem that justifies your answer.

Solution: We have shown that power series can be differentiated term by term thus

$$\begin{aligned} f'(x) &= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \cdots\right)' \\ &= 0 - \frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \frac{8x^7}{9!} - \frac{10x^9}{11!} + \cdots \\ &= -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \frac{8x^7}{9!} - \frac{10x^9}{11!} + \cdots \end{aligned}$$

□

(c) Give the first five terms of the power series for $\int_0^x f(t)\,dt$ and quote a theorem that justifies your answer.

Solution: We have also shown that power series can be integrated term at a time. Therefore

$$\begin{aligned} \int_0^x f(t)\,dt &= \int_0^x \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \frac{t^8}{9!} - \frac{t^{10}}{11!} + \cdots\right) dt \\ &= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} - \frac{x^{11}}{11 \cdot 11!} + \cdots \end{aligned}$$

□

6. Find the sum of the following series:

$$\sum_{n=0}^{\infty} ar^{k+2n} = ar^k + ar^{k+2} + ar^{k+4} + ar^{k+6} + \cdots$$

Where $|r| < 1$.

Solution: This is a geometric series. Thus

$$ar^k + ar^{k+2} + ar^{k+4} + ar^{k+6} + \cdots = \frac{\text{first}}{1 - \text{ratio}} = \frac{ar^k}{1 - r^2}.$$

7. Give the first there nonzero terms of for the Taylor's of $f(x) = (1+x)^{-2}$ about $x = 0$.

Solution: By the binomial theorem we have for $|x| < 1$

$$(1+x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n.$$

The binimial coefficent $\binom{-2}{n}$ is given by

$$\begin{aligned}\binom{-2}{n} &= \frac{(-2)(-3)\cdots(-2-n+1)}{n!} \\ &= \frac{(-2)(-3)\cdots(-n-1)}{n!} \\ &= \frac{(-1)^n(n+1)!}{n!} \\ &= (-1)^n(n+1).\end{aligned}$$

Thus for $|x| < 1$

$$(1+x)^{-2} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - \cdots$$

and therefore the first three nonzero terms are

$$1 - 2x + 3x^2 - \cdots$$

□