Mathematics 555

Test 3

Name: **Answer Key**

1. State the following:

(a) The definition of $\sum_{n=1}^{\infty} f_n = f$ ***uniformly***.

Solution: For all $\varepsilon > 0$ there is a $N > 0$ such that

$$\forall x \left(n > N \quad \implies \quad \left| f(x) - \sum_{k=1}^{n} f_k(x) \right| < \varepsilon \right).$$

☐

(b) The definition of $\langle K_n \rangle_{n=1}^{n}$ being a ***Dirac sequence***.

Solution: $\langle K_n \rangle_{n=1}^{n}$ is a sequence of functions $K_n \colon \mathbf{R} \to \mathbf{R}$ such that

(a) $K_n(x) \geq 0$ for all x.

(b) $\int_{-\infty}^{\infty} K_n(x)\,dx = 1$.

(c) For all $\delta > 0$ the limit

$$\lim_{n\to\infty} \int_{|y|>\delta} K_n(y)\,dy = 0$$

holds. ☐

(c) The ***Weierstrass approximation theorem***.

Solution: If $f \colon [a,b] \to \mathbf{R}$ is a continuous function then there is a sequence of polynomials $p_1, p_2, p_3, \ldots$ such that

$$\lim_{n\to\infty} p_n(x) = f(x)$$

uniformly on $[a,b]$. ☐

(d) The ***Weierstrass*** M***-test***.

Solution: Let $f_1, f_2, f_3, \ldots$ be a sequence of functions on a subset A of $\mathbf{R}$ such that there are constants M_n with

$$|f_n(x)| \leq M_n$$

for all $x \in A$ and

$$\sum_{n=1}^{\infty} M_n < \infty.$$

Then

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly and absolutely on A. ☐

2. (10 points) Give an example of a sequence of functions on the interval $[1, 2]$ that converges pointwise, but not uniformly.

Solution: Note the interval is $[1, 2]$ not $[0, 1]$ and that examples that work on $[0, 1]$ do not necessarily on a different interval. Maybe the easiest example is

$$f_n(x) = (x-1)^n$$

then pointwise

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 1 \le x < 2; \\ 1, & x = 2. \end{cases}$$

but the limit can not be uniform becasue the limit function is not continuous and the uniform limit of continuous functions is continuous. □

Alternate Solution: If $\langle g_n \rangle_{n=1}^{\infty}$ is an example on $[0, 1]$, then

$$f_n(x) = g_n(x-1)$$

will be a solution on $[1, 2]$. We know that

$$g_n(x) = \frac{nx}{1 + n^2x^2}$$

is an example on $[0, 1]$ therefore

$$f_n(x) = \frac{n(x-1)}{1 + n^2(x-1)^2}$$

is an example on $[1, 2]$. □

3. (a) Prove that the series

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(2^k x)}{4^k}$$

converges uniformly on all of $\mathbf{R}$.

Solution: Let $f_k = \sin(2^k x)/4^k$ and $f(x) = \sum_{k=1}^{\infty} f_k(x)$. Then

$$|f_k(x)| = \left| \frac{\sin(2^k x)}{4^k} \right| \le \frac{1}{4^k} = M_k$$

where this defines M_k. But

$$\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{4^k} < \infty$$

so the series for $f(x)$ converges absolutely and uniformly by the Weierstrass M-test. □

(b) Explain briefly (you just need to quote the right theorem) why $f(x)$ is continuous.

The functions $f_k = \sin(2^k x)/4^k$ are continuous and thus so are the the partial sums $S_n = \sum_{k=1}^{n} f_k$. The partial sums converge uniformly to f and the uniform limit of continuous functions is continuous. □

(c) Explain briefly (you need just to say the right thing about the series for the derivative and quote the right theorem) the derivative $f'(x)$ exists.

Solution: Formally the series for the derivative is

$$\sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k}.$$

This again converges uniform by the Weierstrass M-test with $M_k = 1/2^k$. But we have a theorem that if the series $\sum_{k=1}^{\infty} f_k'(x)$ converges uniformly and the series $f = \sum_{k=1} f_k$ converges for at least one point, then f is differentiable and $f' = \sum_{k=1}^{\infty} f_k'(x)$. □

4. Does the sequence of functions $f_n = x^n e^{-nx}$ converge to zero uniformly on $[0,\infty)$? Justify your answer.

Solution: Clearly $f_n(x) \geq 0$ for all x and n. We find the maximum of $f_n(x)$.

$$f_n'(x) = nx^{n-1}x^{n-1}e^{-nx} + x^n\left(-ne^{-nx}\right) = nx^{n-1}e^{-nx}(1-x).$$

Thus $f_n'(x) > 0$ for $0 < x < 1$ and $f_n'(x) < 0$ on $(1,\infty)$. Therefore f_n is increasing on $[0,1]$ and decreasing on $[1,\infty]$. Whence f_n has its maximum at $x = 1$. Thus

$$0 \leq f_n(x) \leq f_n(1) = 1^n e^{-n1} = \frac{1}{e^n}.$$

Thus if $\varepsilon > 0$ and we let N be so that $1/e^N < \varepsilon$ (so $N = \ln(1/\varepsilon)$ will work), then

$$n > N \quad \Longrightarrow \quad |f_n(x) - 0| = f_n(x) \leq \frac{1}{e^n} \leq \frac{1}{e^N} < \varepsilon$$

so the convergence is uniform. □

5. Let f be uniformly continuous on all of $\mathbf{R}$. For $h > 0$ define

$$f_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(y)\,dy.$$

Prove

$$\lim_{h\to 0} f_h(x) = f(x) \quad \text{uniformly.}$$

Solution: Let $\varepsilon > 0$. As f is uniformly continuous there is a $\delta > 0$ such that

$$(1) \qquad |x-y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon \quad \text{for all} \quad x, y \in \mathbf{R}.$$

Let $h < \delta$, then for any $x \in \mathbf{R}$

$$\begin{aligned}
|f(x) - f_h(x)| &= \left| f(x)\cdot 1 - \frac{1}{2h}\int_{x-h}^{x+h} f(y)\,dy \right| \\
&= \left| f(x)\left(\frac{1}{2h}\int_{x-h}^{x+h} 1\,dy\right) - \frac{1}{2h}\int_{x-h}^{x+h} f(y)\,dy \right| \\
&= \left| \frac{1}{2h}\int_{x-h}^{x+h} f(x)\,dy - \frac{1}{2h}\int_{x-h}^{x+h} f(y)\,dy \right| \\
&= \left| \frac{1}{2h}\int_{x-h}^{x+h} f(x) - f(y)\,dy \right| \\
&\leq \frac{1}{2h}\int_{x-h}^{x+h} |f(x) - f(y)|\,dy \\
&< \frac{1}{2h}\int_{x-h}^{x+h} \varepsilon\,dy \qquad \text{(by (1) and } x-\delta < y < x+\delta\text{))} \\
&= \varepsilon.
\end{aligned}$$

Thus for all $x \in \mathbf{R}$

$$h < \delta \quad \implies \quad |f(x) - f_h(x)| < \varepsilon.$$

Therefore $\lim_{h\to 0} f_h(x) = f(x)$ uniformly on $\mathbf{R}$. $\square$

6. Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous functions on $[0,1]$ such that $f_1 \geq f_2 \geq f_3 \geq f_4 \geq \cdots$ and $\lim_{n\to\infty} f_n(x) = 0$ pointwise. Prove that $\lim_{n\to\infty} f_n(x) = 0$ uniformly. *Hint:* Let $\varepsilon > 0$ and let $U_n = \{x \in [0,1] : f_n(x) < \varepsilon\}$. Then U_n is open (we proved this last term and you and use it without proof here). Now show $U_n \subseteq U_{n+1}$ and recall the statement of the Heine-Borel theorem.

Solution: For each $x \in [0,1]$, the sequence $f_1(x), f_2(x), f_3(x), \ldots$ is monotone decreasing to the limit 0. Therefore $f_n(x) \geq 0$ for all x.

Let $\varepsilon > 0$. Then for each $x \in [0,1]$ we have $\lim_{n\to\infty} f_n(x) = 0$ so there is an $n > 0$ such that $f_n(x) < \varepsilon$. Thus each $x \in [0,1]$ is in some U_n. Therefore $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$ is an open cover of $[0,1]$. By the Heine-Borel there is a finite subcover $\mathcal{U}_0 = \{U_{n_1}, U_{n_2}, \ldots, U_{n_m}\}$. Let $N = \max\{n_1, n_2, \ldots, n_m\}$. Then $U_n \subseteq U_{n+1}$ implies $U_{n_j} \subseteq U_N$ for $j = 1, 2, \ldots, m$. As $\{U_{n_1}, U_{n_2}, \ldots, U_{n_m}\}$ is a cover of $[0,1]$ this implies

$$[0,1] \subseteq U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_m} = U_N.$$

But the definition of U_N this implies $0 \leq f_N(x) < \varepsilon$ for all $x \in [0,1]$. But then for all $x \in [0,1]$ and $n \geq N$ we have

$$0 \leq f_n(x) \leq f_N(x) < \varepsilon.$$

Therefore $\lim_{n\to\infty} f_n(x) = 0$ uniformly. $\square$