Riemann Integration.

Recall that we are using the notation $\mathcal{S}[a,b]$ the vector space of all step functions on $[a,b]$ and $\mathcal{R}[a,b]$ for the vector space of Riemann integrable functions on the $[a,b]$.

Proposition 1. *If f is a bounded function on the closed bounded interval $[a,b]$ then f is integrable if and only if all $\varepsilon > 0$ there are step functions $\varphi, \psi \in \mathcal{S}[a,b]$ such that*

$$\varphi \le f \le \psi$$

and

$$\int_a^b (\psi - \varphi)\, dx < \varepsilon.$$

Problem 1. Prove this. *Hint:* We outlined the proof in class. □

To use this we need to be able to construct some step functions that approximate a given bounded function well. Here we need a little bit more notation.

Definition 2. Let $[a,b]$ be a closed bounded interval. Then a ***partition*** of $[a,b]$ is a list of points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. We denote it by $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$. We also use the notation

$$\Delta x_j = x_j - x_{j-1}.$$

(See Figure 1.) □

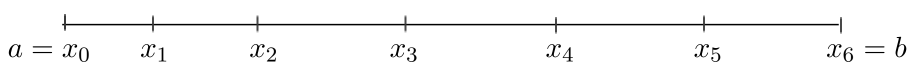

FIGURE 1. A partition of the interval $[a,b]$ into $n = 6$ pieces. The j-th interval $[x_{j-1}, x_j]$ has length $\Delta x_j = x_j - x_{j-1}$.

If f is a monotone increasing function on $[a,b]$ and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ define two step functions by $\varphi_{f,\mathcal{P}}(b) = f(b)$,

$$\varphi_{f,\mathcal{P}}(x) = f(x_{j-1}) \qquad \text{for} \qquad x \in [x_{j-1}, x_j)$$

and $\psi_{f,\mathcal{P}}(b) = f(b)$

$$\psi_{f,\mathcal{P}} = f(x_j) \qquad \text{for} \qquad x \in [x_{j-1}, x_j).$$

See Figure 2

Proposition 3. *If f is monotone increasing on $[a,b]$ then for any partition, $\mathcal{P}$, of $[a,b]$, with the notation above,*

$$\varphi_{f,\mathcal{P}} \le f \le \psi_{f,\mathcal{P}}$$

on $[a,b]$.

Problem 2. Prove this. □

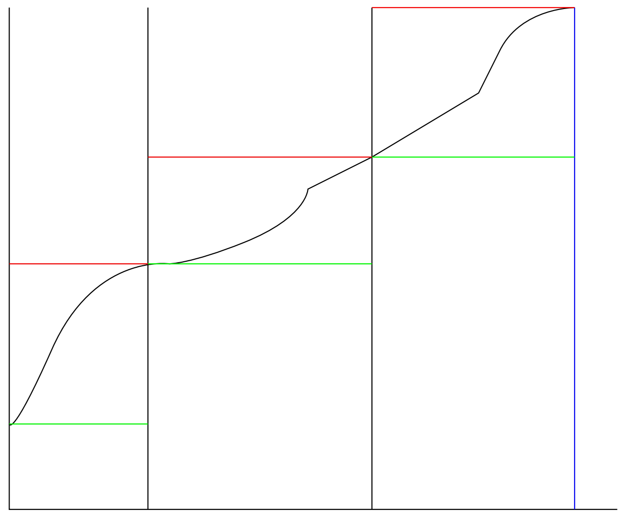

FIGURE 2. A monotone increasing function on $[a,b]$ and a partition, $\mathcal{P}$, with $n=3$ showing the lower step function $\varphi_{f,\mathcal{P}}$ (in green) and the upper step function $\psi_{f,\mathcal{P}}$ (in red).

Definition 4. Given a positive integer n and a closed bounded interval $[a,b]$ the ***uniform partition*** of $[a,b]$ into n sub-intervals is the partition $\mathcal{P}=\{x_0,x_1,\ldots,x_n\}$ with

$$x_j = a + j\left(\frac{b-a}{n}\right)$$

for $j=0,1,\ldots,n$. Note in this case all the lengths, Δx_j of the sub-intervals $[x_{j-1},x_j]$ have the same value $\Delta x = \Delta x_j = (b-a)/n$. □

Now let us consider the monotone increasing function f on the interval $[a,b]$ with the uniform partition, $\mathcal{P}$, of $[a,b]$ with $n=4$. Then $\Delta x = \Delta x_j = (b-a)/4$ and $\varphi_{f,\mathcal{P}} \le f \le \psi_{f,\mathcal{P}}$. Also

$$\int_a^b \varphi_{f,\mathcal{P}}(x)\,dx = \big(f(x_0)+f(x_1)+f(x_2)+f(x_3)\big)\Delta x$$

and

$$\int_a^b \psi_{f,\mathcal{P}}(x)\,dx = \big(f(x_1)+f(x_2)+f(x_3)+f(x_4)\big)\Delta x.$$

Thus

$$\int_a^b \left(\psi_{f,\mathcal{P}}(x) - \psi_{f,\mathcal{P}}(x)\right)\,dx = \left(f(x_4)-f(x_0)\right)\Delta x = \left(f(b)-f(a)\right)\Delta x$$

There is nothing special about $n=4$ in this:

Problem 3. Show that if f is monotone increasing on $[a,b]$, n is a positive integer and $\mathcal{P}=\{x_0,x_1,\ldots,x_n\}$ is the uniform partition of $[a,b]$ into n

sub-intervals, then, with the notation above,

$$\int_a^b (\psi_{f,\mathcal{P}}(x) - \varphi_{f,\mathcal{P}}(x))\, dx = (f(b) - f(a))\,\Delta x = \frac{(f(b) - f(a))(b-a)}{n}. \qquad \square$$

Theorem 5. *If f is a monotone function on the closed bounded interval $[a, b]$, then f is integrable on $[a, b]$.*

Problem 4. Prove this. *Hint:* With out loss of generality assume f is monotone increasing (if f is monotone decreasing replace f by $-f$). Let $\varepsilon > 0$ and let n be a positive integer such that

$$\frac{(f(b) - f(a))(b-a)}{n} < \varepsilon$$

and use Proposition 1 and the last problem. $\square$

Theorem 6. *Let f be a continuous function on $[a, b]$. Then f is integrable on $[a, b]$.*

Proof. Let $\varepsilon > 0$. As f is continuous on the closed bounded set $[a, b]$ it is uniformly continuous on $[a, b]$. Thus there is an $\delta > 0$ such that for $x, y \in [a, b]$.

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

Let n be a positive integer such that

$$\frac{b-a}{n} = \Delta x < \delta$$

and let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be the uniform partition of $[a, b]$ into n sub-intervals. Set

$$\begin{aligned} m_j &= \inf\{f(x) : x \in [x_{j-1}, x_j]\} = \min\{f(x) : x \in [x_{j-1}, x_j]\}, \\ M_j &= \sup\{f(x) : x \in [x_{j-1}, x_j]\} = \max\{f(x) : x \in [x_{j-1}, x_j]\} \end{aligned}$$

where the infimum is achieved as a minimum and the supremum is achieved as a maximum because continuous functions on closed bounded sets achieve their maximums and minimums. Define step functions φ and ψ on $[a, b]$ $\varphi(b) = \psi(b) = f(b)$ and

$$\begin{aligned} \varphi(x) &= m_j \quad \text{for} \quad x_{j-1} \le x < x_j \\ \psi(x) &= M_j \quad \text{for} \quad x_{j-1} \le x < x_j. \end{aligned}$$

Then

$$\varphi \le f \le \psi$$

and

$$\int_a^b (\varphi(x) - \psi(x))\, dx = \sum_{j=1}^{n} (M_j - m_j) \left(\frac{b-a}{n}\right).$$

As f is continuous on the closed bounded interval $[x_{j-1}, x_j]$, f achieves its maximum and minimum on this interval. Thus there are $\alpha_j, \beta_j \in [x_{j-1}, x_j]$

with $f(\alpha_j) = m_j$ and $f(\beta_j) = M_j$. But then $|\alpha_j - \beta_j| \le \Delta x < \delta$ and therefore

$$M_j - m_j = |f(\beta_j) - f(\alpha_j)| < \frac{\varepsilon}{b-a}.$$

Thus

$$\int_a^b (\varphi(x) - \psi(x))\,dx = \sum_{j=1}^{n} (M_j - m_j)\left(\frac{b-a}{n}\right) < \sum_{j=1}^{n} \frac{\varepsilon}{b-a}\left(\frac{b-a}{n}\right) = \varepsilon$$

and the result now follows from Proposition 1. □

Let us record a few more basic facts about integrable functions.

Proposition 7. *If* $f \in \mathcal{R}[a,b]$ *then so is* $g = \max\{f, 0\}$.

Proof. Let $\varepsilon > 0$ Let φ and ψ be step functions on $[a,b]$ such that $\varphi \le f \le \psi$ and $\int_a^b (\psi - \varphi)\,dx < \varepsilon$. Then

$$\varphi_0 = \max\{0, \varphi\}, \qquad \psi_0 = \max\{0, \psi\}$$

are step functions, $\varphi_0 \le \max\{f, 0\} \le \psi_0$ and $0 \le \psi_0 - \varphi_0 \le \psi - \varphi$. Thus

$$\int_a^b (\psi_0 - \varphi_0)\,dx \le \int_a^b (\psi - \varphi)\,dx < \varepsilon$$

and so $\max\{f, 0\}$ is integrable by Proposition 1. □

This implies a good deal more because of the following elementary result.

Lemma 8. *For real numbers* a, b *the following hold*

$$\begin{aligned} \min\{a, 0\} &= -\max\{-a, 0\}, \\ |a| &= \max\{a, 0\} + \max\{-a, 0\}, \\ \max\{a, b\} &= a + \max\{0, b-a\}, \\ \min\{a, b\} &= a + \min\{0, b-a\}. \end{aligned}$$

Proof. Left to reader (and you don't have to turn these in). We did enough of this type of thing last term that I believe you can do it. □

Proposition 9. *If* f *and* g *are integrable on* $[a,b]$ *then so are* $|f|$, $\min\{f, g\}$ *and* $\max\{f, g\}$.

Proof. This follows easily from Proposition 7 and Lemma 8. □

Lemma 10. *If* f *is integrable on* $[a,b]$ *then so is* f^2.

Problem 5. Prove this. *Hint:* As $f^2 = |f|^2$ and $|f|$ is also integrable by replacing f by $|f|$ we can assume $f \ge 0$. As f is integrable it is bounded, say $0 \le f \le B$ on $[a,b]$. Also as f is integrable on $[a,b]$ for $\varepsilon > 0$ there is are step functions φ, ψ such that

$$\varphi \le f \le \psi$$

and

$$\int_a^b (\psi - \varphi)\,dx < \frac{\varepsilon}{2B}.$$

By replacing φ by $\max\{0, \varphi\}$ and ψ by $\min\{\psi, B\}$ we can assume $0 \le \varphi$ and $\psi \le B$. Then φ^2 and ψ^2 are step functions and

$$\varphi^2 \le f^2 \le \psi^2$$

and

$$0 \le \psi^2 - \varphi^2 = (\psi + \varphi)(\psi - \varphi) \le (\psi + \psi)(\psi - \varphi) \le (B + B)(\psi - \varphi).$$

You should now be able to show

$$\int_a^b (\psi^2 - \varphi^2)\,dx < \varepsilon$$

so that Proposition 1 applies. □

Proposition 11. *If f and g are integrable on $[a, b]$ then so is the product fg.*

Problem 6. Prove this. *Hint:* Show

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}$$

and use Lemma 10. □

Proposition 12. *If $a < b < c$ and f is integrable on $[a, c]$ then the restrictions $f\big|_{[a,b]}$ and $f\big|_{[b,c]}$ are integrable on $[a, b]$ and $[b, c]$ respectively and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof. We have shown for any bounded function on $[a, c]$ that

$$\overline{\int_a^c} f(x))\,dx = \overline{\int_a^b} f(x)\,dx + \overline{\int_b^c} f(x)\,dx,$$
$$\underline{\int_a^c} f(x))\,dx = \underline{\int_a^b} f(x)\,dx + \underline{\int_b^c} f(x)\,dx.$$

As f is integrable on $[a, c]$

$$\begin{aligned}\int_a^c f(x)\,dx &= \overline{\int_a^c} f(x)\,dx \\ &= \underline{\int_a^c} f(x)\,dx \\ &= \underline{\int_a^b} f(x)\,dx + \underline{\int_b^c} f(x)\,dx \\ &\le \overline{\int_a^b} f(x)\,dx + \overline{\int_b^c} f(x)\,dx \\ &= \overline{\int_a^c} f(x)\,dx \\ &= \int_a^c f(x)\,dx.\end{aligned}$$

Thus equality must hold at all the intermediate inequalities. Therefore

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx \quad \text{and} \quad \underline{\int_b^c} f(x)\,dx = \overline{\int_b^c} f(x)\,dx$$

which implies the restrictions $f\big|_{[a,b]}$ and $f\big|_{[b,c]}$ are integrable. The rest follows from

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx \quad \text{and} \quad \int_b^c f(x)\,dx = \overline{\int_b^c} f(x)\,dx$$

and that equality holds in the displayed inequality. □

Proposition 13. *Let f be integrable on $[a, b]$ and let $[\alpha, \beta] \subseteq [a, b]$. The f is integrable on $[\alpha, \beta]$.*

Problem 7. Prove this. *Hint:* $[\alpha, \beta] = [a, \beta] \cap [\alpha, b]$ and Proposition 12. □

It is useful to define $\int_a^b f(x)\,dx$ even in the cases where $a = b$ and $b < a$.

Definition 14. For any function f define

$$\int_a^b f(x)\,dx = 0.$$

If $b < a$ and f is integrable on $[b, a]$ define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

□

Proposition 15. *If f is integrable on the interval $[x_1, x_2]$ and $a, b, c \in [x_1, x_2]$ then, with the definitions above,*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof. This is just checking case by case (i.e. $a \le b \le c$, $a \le c \le b$ etc.) and is left to the reader. And please do not hand it in. $\square$

Proposition 16. *Let $f(x)$ be integrable on $[a,b]$ and let $F\colon [a,b] \to \mathbf{R}$ be defined by*

$$F(x) = \int_a^x f(t)\,dt$$

then there is a constant M such that

$$|F(x_2) - F(x_1)| \le M|x_2 - x_1|$$

and therefore F is continuous on $[a,b]$.

Problem 8. Prove this. *Hint:* As f is integrable on $[a,b]$, it is bounded on $[a,b]$, say $|f(x)| \le M$ on $[a,b]$. Without loss of generality we can assume that $x_1 \le x_2$. Then

$$|F(x_2) - F(x_1)| = \left|\int_a^{x_2} f(t)\,dt - \int_a^{x_1} f(t)\,dt\right| = \left|\int_{x_1}^{x_2} f(t)\,dt\right| \le \int_{x_1}^{x_2} |f(t)|\,dt$$

and it should be easy from here. $\square$

Theorem 17 (Fundamental Theorem of Calculus Form 1)**.** *Let f be integrable on $[a,b]$. Define new function $F\colon [a,b] \to \mathbf{R}$ by*

$$F(x) = \int_a^x f(t)\,dt.$$

If f is continuous at the point $x \in (a,b)$, then the derivative of F exists at x and

$$F'(x) = f(x).$$

Problem 9. Prove this. *Hint:* First note

$$1 = \frac{1}{h}\int_x^{x+h} 1\,dt.$$

Multiply by $f(x)$ to get

$$f(x) = \frac{1}{h}\int_x^{x+h} f(x)\,dt$$

Also note

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

Combining some of these formulas we get

$$\begin{aligned}
\frac{F(x+h) - F(x)}{h} - f(x) &= \frac{1}{h}\int_x^{x+h} f(t)\,dt - \frac{1}{h}\int_x^{x+h} f(x)\,dt \\
&= \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt.
\end{aligned}$$

Let $\varepsilon > 0$. As f is continuous at x there is a $\delta > 0$ such that

$$|t - x| < \delta \quad\Longrightarrow\quad |f(t) - f(x)| < \varepsilon.$$

Put this all together to show

$$|h| < \delta \qquad \implies \qquad \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| < \varepsilon$$

and explain why this shows $F'(x) = f(x)$. □

Theorem 18 (Fundamental Theorem of Calculus Forn 2)**.** *Let f be continuous on $[a,b]$ and let F be continuous on $[a,b]$ and differentiable (a,b) with $F' = f$ on (a,b). Then*

$$\int_a^b f(t)\,dt = F(b) - F(a) = F\Big|_a^b.$$

Problem 10. Prove this. *Hint:* Let

$$G(x) = \int_a^x f(t)\,dt - F(x)$$

and show $G'(x) = 0$ for $x \in (a,b)$. □

Corollary 19. *If f is continuous on $[a,b]$ and F is any anti-derivative of f on $[a,b]$ (that is $F'(x) = f(x)$ for $x \in [a,b]$), then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Problem 11. Prove this. □

Definition 20. Let f be integrable on $[a,b]$. Then the ***average value*** of f on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$ □

Theorem 21 (The First Mean Value Theorem for Integrals)**.** *If f is continuous on $[a,b]$, then it achieves its average value. That is there is a $\xi \in (a,b)$ with*

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Problem 12. Prove this. *Hint:* As f is continuous on the closed bounded set $[a,b]$, it achieves its maximum and minimum on this interval. Let $m = \min\{f(x) : x \in [a,b]\}$ and $M = \max\{f(x) : x \in [a,b]\}$ and let $\alpha, \beta \in [a,b]$ such that $f(\alpha) = m$ and $f(\beta) = M$. Now

$$f(\alpha) = m = \frac{1}{b-a}\int_a^b m\,dx \le \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$f(\beta) = M = \frac{1}{b-a}\int_a^b M\,dx \ge \frac{1}{b-a}\int_a^b f(x)\,dx$$

and recall the intermediate value theorem. □