Here are some problems that did not make in onto the test, or supplement some topics we have done in class.

Problem 1. Let $\lim_{n\to\infty} f_n \to f$ uniformly on $[a,b]$ where each f_n is continuous. Let $\langle x_k\rangle_{k=1}^{\infty}$ be a sequence with $\lim_{k\to\infty} x_k = x$. Prove

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

□

Problem 2. Give an example of a sequence of functions with $\lim_{n\to\infty} f_n = 0$ pointwise on $[0,1]$ and a sequence $\langle x_k\rangle_{k=1}^{\infty}$ with $\lim_{k\to\infty} x_k = 0$, but $\lim_{n\to\infty} f_n(x_n) = 1$. □

Theorem 1 (Dini's Theorem)**.** *Let $\langle f_n\rangle_{k=1}^{\infty}$ be a sequence of continuous functions that converges pointwise and monotonically to the continuous function f. Then $\lim_{n\to\infty} f_n = f$ uniformly.* □

Problem 3. Prove this. *Hint:* On the test you proved that if $\langle g_n\rangle_{k=1}^{\infty}$ is a sequence of continuous functions that converges pointwise and monotonically to 0 on a closed bounded interval, then $\lim_{n\to\infty} g_n = 0$ uniformly. Reduce the general case to this. □

Problem 4. Let f be a continuous function on $[a,b]$ such that

$$\int_a^b f(x)^2\,dx = 0.$$

Prove $f(x) = 0$ for all $x \in [a,b]$. □

Problem 5. Let f, p be continuous functions on $[a,b]$ and let $p_1, p_2, p_3, \ldots$ be a sequence of continuous functions such that $p_n \to p$ uniformly. Then show $fp_n \to fp$ uniformly. *Hint:* As f is continuous on a closed bounded set $[a,b]$ it is bounded. □

Problem 6. Let f be continuous on $[a,b]$ and assume that

$$\int_a^b x^n f(x)\,dx = 0$$

for all $x = 0, 1, 2, \ldots$. Show that $f(x) = 0$ for all $x \in [a,b]$. □