Review for Test 1.

The first topic we covered this term was limits of sequences. The basic definitions are results are summed up on the handout we had on the subject (labeled Homework due Wednesday January 30 on the class web page and here after refereed to as Sequence Notes). In particular know the definitions. This includes the following

- The definition of $\lim_{n\to\infty} a_n = A$ (this includes knowing the meaning of convergent, divergent etc.) And how to compute limits. Some examples
  (a) $\displaystyle\lim_{n\to\infty} \frac{13^n}{n!} = 0.$
  (b) $\displaystyle\lim_{n\to\infty} \left((n^3+n)^{2/3} - n^2\right) = \frac{2}{3}.$
  (c) If $\lim_{n\to\infty} a_n = A$, then $\displaystyle\lim_{n\to\infty} \frac{a_n + 2a_{n+1} + 2a_{n+2}}{5} = A$. *Hint:*
  $$\frac{a_n + 2a_{n+1} + 2a_{n+2}}{5} - A = \frac{(a_n - A) + 2(a_{n+1} - A) + 2(a_{n+2} - A)}{5}.$$
- The definition of $\limsup_{n\to\infty} a_n$, $\liminf_{n\to\infty} a_n$ and how to compute them in easy cases.
- Definitely be able to prove Theorems 10 and 12 in Sequence Notes.
- Know the definition of subsequence and the statement of the Bolzano-Weierstrass Theorem. Be able to use the Bolzano-Weierstrass Theorem to be able to do proofs such as showing that a continuous function on a closed bounded set achieves its maximum and minimum (Theorem 27 in Sequence Notes).
- You will almost certainly be ask to give the definition of a Cauchy sequence. You should also be able to show that a convergent sequence is Cauchy.

The next topic we covered is Riemann integration. Some of this is in the notes on the web page titled Homework due on Wednesday, February 12 and here after referred as Integration Notes.

- First off know the definition of a step function $\varphi = \sum_{j=1}^n a_j \chi_{I_j}$ and its integral
  $$\int \varphi(x)\,dx = \sum_{j=1}^n a_j |I_j|$$

where $|I_j|$ is the length of the interval I_j.

- If f is a bounded function on $[a,b]$ know the definition of the upper and lower integrals.

$$\underline{\int_a^b} f(x)\,dx, \qquad \overline{\int_a^b} f(x)\,dx.$$

and the definition of when f is integrable and the definition of $\int_a^b f(x)\,dx$.

- The basic properties of the integral such as

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

That f is linear.

- The two basic existence theorems for the integral. That is that monotone functions and continuous on closed bounded intervals are integrable (Theorems 5 and 6 in INTEGRATION NOTES). Some sample problems
  (a) Explain why the function

$$f(x) = \begin{cases} x^2 3^{x^2}, & 0 \le x \le 1; \\ -5 - x^2, & 1 < x \le 2. \end{cases}$$

  is integrable.

- The two forms of the Fundamental Theorem of Calculus. Here are some sample problems:
  (a) If f is a continuous function on $\mathbf{R}$

$$\frac{d}{dx}\int_a^{x^2} f(t)\,dt = 2xf(x^2)$$

  (b) If f is a continuous function on $\mathbf{R}$

$$\frac{d}{dx}\int_x^{2x} f(t)\,dt = 2f(2x) - f(x).$$

  Hint:

$$\int_x^{2x} f(t)\,dt = \int_0^{2x} f(t)\,dt - \int_0^x f(t)\,dt.$$

  (c) If f is continuous on $[a,b]$, and $y(x)$ is defined by

$$y(x) = e^x \int_a^x e^{-t} f(t)\,dt,$$

then satisfies

$$y'(x) - y(x) = f(x)$$

on $[a, b]$. *Hint:* Compute $y'(x)$ by first using the product rule.

Finally we have talked about approximating integrals. If $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$ and $\xi_j \in [x_{j-1}, x_j]$, then the sum

$$\sum_{j=1}^{n} f(\xi_j)\Delta x_j$$

approximates $\displaystyle\int_a^b f(x)\,dx$. Here are some examples of this

Problem. Compute the following by interpreting them as limits of Riemann sums.

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^{3/2}} \sum_{j=1}^{n} \sqrt{j}$.

(b) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{j=-n+1}^{n} \cos(j\pi/2)$.

Problem. Let $f\colon [a, b] \to \mathbf{R}$ satisfy $|f(x) - f(y)| \le 7|x - y|$ and let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be the partition of $[a, b]$ into n uniform sub-intervals. The for any selection $\langle \xi_j \rangle_{j=1}^{n}$ for this partition

$$\left| \int_a^b f(x)\,dx - \sum_{j=1}^{n} f(\xi_j)\Delta_j \right| \le \frac{7(b-a)^2}{n}.$$

□

And of course there will be the usual surprise mystery questions.