Some basics about sequences.

We need to look at anther type of limit. Hopefully this will be close enough to the other limits we have looked at to be easy. Recall that $\mathbf{Z}$ is the set of integers.

Definition 1. A ***sequence*** of real numbers is a function $a\colon \{n : n \geq k \text{ and } n \in \mathbf{Z}\} \to \mathbf{R}$ where k is some integer. $\square$

We will usually write the value of $a(n)$ as a_n and the sequence as $\langle a_n \rangle_{n=k}^{\infty}$. For example

$$\left\langle \frac{1}{n+1} \right\rangle_{n=1}^{\infty}$$

is the sequence with $a_n = \frac{1}{n+1}$ and $k = 1$. Given a sequence in the intuitive sense, that as a list of numbers, it can be made into a sequence in our technical in more than one way. For example the sequence

$$1, 4, 9, 16, 25, 36, 49, \ldots$$

would most naturally be written as

$$\langle n^2 \rangle_{n=1}^{\infty}$$

but could also be expressed as

$$\left\langle (n+1)^2 \right\rangle_{n=0}^{\infty}, \quad \left\langle (n-1)^2 \right\rangle_{n=2}^{\infty}, \qquad \left\langle (n+10)^2 \right\rangle_{n=-9}^{\infty}.$$

Also there is nothing special about using n as the index. Thus

$$\langle a_n \rangle_{n=1}^{\infty} = \langle a_k \rangle_{k=1}^{\infty} = \langle a_t \rangle_{t=1}^{\infty}.$$

This notation differs from that in the text, where he writes $\{a_n\}_{n=k}^{\infty}$ rather than $\langle a_n \rangle_{n=k}^{\infty}$. I prefer the notation used here as it makes clear the difference between the sequence $\langle a_n \rangle_{n=k}^{\infty}$ from the set $\{a_k, a_{k+1}, \ldots\}$ To take an extreme example, let $\langle a_n \rangle_{n=1}^{\infty}$ be the constant sequence with $a_n = 1$ for all n. Then the set $\{a_1, a_2, \ldots\}$ is just the one element set $\{1\}$ which is different from the infinite sequence of ones $\langle 1, 1, 1, \ldots \rangle$.

We will also define a sequence by just saying what its n-th term is. For example saying "the sequence defined by $a_n = 1/(n^2+1)$ for $n \geq 0$" which is just $\left\langle 1/(n^2+1) \right\rangle_{n=0}^{\infty}$.

And now to the analysis part, which is to say limits.

Definition 2. The limit of the sequence $\langle a_n \rangle_{n=k}^{\infty}$ is A, written as

$$\lim_{n\to\infty} a_n = A,$$

iff for all $\varepsilon > 0$ there is a $N = N_\varepsilon$ such that

$$n > N \qquad \Longrightarrow \qquad |a_n - A| < \varepsilon.$$

When the limit of $\langle a_n \rangle_{n=k}^{\infty}$ exists we say it that it ***converges***, or that it is is ***convergent***. $\square$

Here is some practice in working with this definition

Problem 1. Directly from the definition show
(a) $\lim_{n\to\infty} \frac{n+1}{n} = 1$,
(b) $\lim_{n\to\infty} (-1)^n$ does not exist.

We now just follow the same pattern we did with limits of functions of a real variable.

Proposition 3. *The limit of a sequence is unique.*

Proof. This is we need to show that if $\lim_{n\to\infty} = A$ and $\lim_{n\to\infty} a_n = A_1$ then $A = A_1$. Towards a contradiction, assume $A \neq A_1$. Let $\varepsilon = |A - A_1|/2$. Then there is a $N > 0$ such that

$$n \geq N \quad \implies |a_n - A| < \varepsilon = \frac{|A - A_1|}{2}$$

and an N_1 with

$$n \geq N_1 \quad \implies |a_n - A_1| < \varepsilon = \frac{|A - A_1|}{2}.$$

Let $n > \max\{N, N_1\}$. Then

$$\begin{aligned} |A - A_1| &= |A - a_n - (A_1 - a_n)| \\ &\leq |A - a_n| + |A_1 - a_n| \\ &< \frac{|A - A_1|}{2} + \frac{|A - A_1|}{2} \\ &= |A - A_1|. \end{aligned}$$

That is $|A - A_1| < |A - A_1|$ which is a contradiction and completes the proof. □

Theorem 4. *If* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$, *then*

$$\lim_{n\to\infty} (a_n + b_n) = A + B.$$

Problem 2. Prove this. *Hint:* Let $\varepsilon > 0$ and let N_1 and N_2 be such that

$$n \geq N_1 \quad \implies \quad |a_n - A| < \frac{\varepsilon}{2} \qquad \text{and} \qquad n \geq N_2 \quad \implies \quad |b_n - B| < \frac{\varepsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$. If $n > N$ use the triangle inequality on $|(a_n + b_n) - (A + B)|$. □

Definition 5. If a sequence $\langle a_n \rangle_{n=k}^{\infty}$ is ***bounded*** iff there is a constant M such that $|a_n| \leq M$ for all $n \geq k$. □

Theorem 6. *If a sequence* $\langle a_n \rangle_{n=k}^{\infty}$ *is convergent, then it is bounded.*

Problem 3. Prove this. *Hint:* Let $A = \lim_{n\to\infty} a_n$ and let $\varepsilon = 1$. Then there is a $N > 0$ such that

$$n > N \quad \implies |a_n - A| < \varepsilon = 1.$$

Set

$$M = \max\{|A| + 1, |a_k|, |a_{k+1}|, |a_{k+2}|, \ldots, |a_n|\}$$

and show that $|a_n| \le M$ for all n. □

Theorem 7. *If* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$*, then*

$$\lim_{n\to\infty} a_n b_n = AB.$$

Problem 4. Prove this. *Hint:* Note that by doing the adding and subtracting trick we have

$$|a_n b_n - AB| = |a_n(b_n - B) + (a_n - A)B| \le |a_n||b_n - B| + |a_n - A||B|.$$

But we know that the sequence $\langle a_n \rangle_{n=k}^{\infty}$ is convergent and therefore by Theorem 6 there is a constant M such that $|a_n| \le M$ for all n. Thus

$$|a_n b_n - AB| \le M|b_n - B| + |a_n - A||B|.$$

and you can take it from here. □

Here is natural a natural generalization of Theorem 4.

Theorem 8. *If* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$*, and* α *and* β *are constants then*

$$\lim_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha A + \beta B.$$

Problem 5. Prove this. □

Theorem 9. *Let* I *be an open interval and* $f\colon I \to \mathbf{R}$ *a function that is continuous on* I*. Let* $\langle a_n \rangle_{n=1}^{\infty}$ *be a sequence with* $a_n \in I$ *for all* $n \ge k$*. Then if* $\langle a_n \rangle_{n=1}^{\infty}$ *is convergent and* $\lim_{n\to\infty} a_n = A \in I$*, then* $\langle f(a_n) \rangle_{n=k}^{\infty}$ *is also convergent and*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

Problem 6. Prove this. *Hint:* We wish to show $\lim_{n\to\infty} f(a_n) = f(A)$. Let $\varepsilon > 0$. As f is continuous at A there is a $\delta > 0$ such that

$$|x - A| < \delta \qquad \Longrightarrow \qquad |f(x) - f(A)| < \varepsilon.$$

As $\lim_{n\to\infty} a_n = A$ there is a N such that

$$n > N \qquad \Longrightarrow \qquad |a_n - A| < \delta,$$

and the rest is up to you. □

More generally we have

Theorem 10. *Let* f *be defined in a neighborhood of* x_0*. The the following are equivalent*

(a) f *is continuous at* x_0*.*
(b) *For every sequence with* $\lim_{n\to\infty} a_n = x_0$ *we have* $\lim_{n\to\infty} f(a_n) = f(x_0)$*.*

Proof. (a) $\implies$ (b). This is just an easy variant on the proof of Theorem 9 and is left to you. (You do not have to hand it in.)

(b) $\implies$ (a). Assume, towards a contradiction, that (b) holds, but that f is not continuous at x_0. This means that there is a $\varepsilon > 0$ such that for all $\delta > 0$ there is a x with

$$|x - x_0| < \delta \qquad \text{and} \qquad |f(x) - f(x_0)| \geq \varepsilon.$$

In particular each positive integer n we let $\delta = 1/n$ to find a number a_n with

$$|a_n - x_0| < \frac{1}{n} \qquad \text{and} \qquad |f(a_n) - f(x_0)| \geq \varepsilon.$$

Then

$$\lim_{n\to\infty} a_n = x_0$$

(for if $N = 1/\varepsilon$ and $n > N$, then $|a_n - x_0| < 1/n < 1/N = \varepsilon$). But

$$|f(a_n) - f(x_0)| \geq \varepsilon$$

for all n. Thus $\lim_{n\to\infty} f(a_n) \neq f(x_0)$ which contradicts that (b) holds. This completes the proof. □

Definition 11. The sequence $\langle a_n\rangle_{n=k}^{\infty}$ is ***monotone increasing*** iff $a_n \leq a_{n+1}$ for all $n \geq k$. It is ***monotone decreasing*** iff $a_n \geq a_{n+1}$ for all $n \geq k$. It is ***monotone*** iff it is either monotone increasing or monotone decreasing. □

Theorem 12. *A bounded monotone sequence is convergent.*

Problem 7. Prove this. *Hint:* Let $\langle a_n\rangle_{n=k}^{\infty}$ be the sequence. Without loss of generally we can assume that $\langle a_n\rangle_{n=k}^{\infty}$ is monotone increasing (otherwise replace $\langle a_n\rangle_{n=k}^{\infty}$ by $\langle -a_n\rangle_{n=k}^{\infty}$). As $\langle a_n\rangle_{n=k}^{\infty}$ is bounded the supremum of the set $\{a_n : n \geq k\} = \{a_k, a_{k+1}, a_{k+2}, \ldots\}$ exists. Let

$$A = \sup\{a_n : n \geq k\} = \sup\{a_k, a_{k+1}, a_{k+2}, \ldots\}.$$

Let $\varepsilon > 0$. By the definition of the supremum there is a N such that

$$A - \varepsilon < a_N \leq A.$$

You take it from here. □

Lemma 13. *Let $\langle a_n\rangle_{n=k}^{\infty}$ be a bounded sequence and let*

$$\overline{A}_n := \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$
$$\underline{A}_n := \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

(a) *$\langle \overline{A}_n\rangle_{n=k}^{\infty}$ is bounded and monotone decreasing, and*
(b) *$\langle \underline{A}_n\rangle_{n=k}^{\infty}$ is bounded and monotone increasing.*

Problem 8. Prove part (a) of the last lemma. □

In light of this lemma and Theorem 12 the following limits exists for any bounded sequence.

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$
$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

Problem 9. Compute the following (you do not have to prove your answers)

$$\limsup_{n\to\infty}(-1)^n(1+1/n), \qquad \liminf_{n\to\infty}(-1)^n(1+1/n).$$ □

Recreational Extra Credit Problem. Assuming that π is irrational show

$$\limsup_{n\to\infty}\sin(n) = 1, \qquad \liminf_{n\to\infty}\sin(n) = -1.$$

Hint: Show numbers of the form $n + 2m\pi$ with m and n integers are dense in the real numbers. □

Lemma 14. *If $A = \lim_{n\to\infty} a_n$ exists, then*

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = A.$$

Proof. We did this in class. □

Proposition 15. *Let $\langle a_n\rangle_{n=k}^{\infty}$ and $\langle b_n\rangle_{n=k}^{\infty}$ be bounded sequences. Then*

(a) *If $a_n \le b_n$ for all n then*

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n \qquad \textit{and} \qquad \limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n$$

(b)

$$\liminf_{n\to\infty}(a_n + b_n) \ge \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

(c)

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

Problem 10. Prove the first inequality of (a) and the inequality of part (c).

Theorem 16. *Let $\langle a_n\rangle_{n=k}^{\infty}$ be a bounded sequence. Then it is convergent if and only if*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Proof. In light of Lemma 14 we only need show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ implies that the sequence is convergent. Let

$$A = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

and let $\varepsilon > 0$. Buy the definitions of lim inf and lim sup there are $N_1, N_2 > 0$ such that

$$n \ge N_1 \qquad \Longrightarrow \qquad |A - \inf\{a_n, a_{n+1}, \ldots\}| < \varepsilon,$$
$$n \ge N_2 \qquad \Longrightarrow \qquad |A - \sup\{a_n, a_{n+1}, \ldots\}| < \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$A - \varepsilon < \inf\{a_n, a_{n+1} \leq a_n \leq \sup\{a_n, a_{n+1} < A + \varepsilon$$

and thus

$$n > N \qquad \Longrightarrow \qquad |a_n - A| < \varepsilon.$$

Thus $\lim_{n\to\infty} a_n = A$ as required. □

Definition 17. The sequence $\langle a_n \rangle_{n=k}^{\infty}$ is ***Cauchy*** iff for all $\varepsilon > 0$ there is a N such that

$$n, m \geq N \qquad \Longrightarrow \qquad |a_m - a_n| < \varepsilon.$$

□

Lemma 18. *If $\langle a_n \rangle_{n=k}^{\infty}$ is convergent, then it is a Cauchy sequence.*

Problem 11. Prove this.

Lemma 19. *If $\langle a_n \rangle_{n=k}^{\infty}$ is a Cauchy sequence, then it is bounded.*

Problem 12. Prove this. *Hint:* We outlined the proof in class.

Theorem 20. *The sequence $\langle a_n \rangle_{n=k}^{\infty}$ is convergent if and only if is a Cauchy sequence.*

Problem 13. Prove this. *Hint:* In light of the last two lemmata[1] we only need to show that if the sequence is Cauchy, then it is convergent. To do this we only need show

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

and the proof of this was outlined in class.

Problem 14. Problems 33 and 34 on Page 194 of the text.

Definition 21. Let $\langle a_n \rangle_{n=1}^{\infty}$. Then a ***subsequence*** of a $\langle a_n \rangle_{n=1}^{\infty}$ is a sequence of the form $\langle b_k \rangle_{k=1}^{\infty}$ where $b_k = a_{n_k}$ and the integers n_k satisfy $n_k < n_{k+1}$. □

The following is elementary

Proposition 22. *If $\langle a_n \rangle_{n=1}^{\infty}$ is a convergent sequence, say $\lim_{n\to\infty} a_n = A$, then every subsequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$ also converges to A.*

Proof. Let $\varepsilon > 0$. Then there is N such that $n > N$ implies $|a_n - A| < \varepsilon$. Then there is a positive integer K such that $n_K > N$. Then $k > K$ implies $|a_{n_k} - A| < \varepsilon$. □

Lemma 23. *Every sequence $\langle a_n \rangle_{n=1}^{\infty}$ has a monotone subsequence.*

[1] "lemmata" is the Greek form of the plural of "lemma".

Proof. Call a the term a_n a ***peak*** of the sequence if $a_n > a_m$ for all $m > n$. That is if a_n is larger than all the terms that follow it. If there are infinitely many peaks, say $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ with $n_1 < n_2 < n_3 \cdots$. Then $\langle a_{n_k} \rangle_{k=1}^{\infty}$ is a strictly decreasing subsequence and we are done.

So assume that there are only finitely many peaks. Let a_ℓ be the last peak and let $n_1 = \ell + 1$. Then, as a_{n_1} is not a peak, there is a $n_2 > n_1$ with $a_{n_2} \geq a_{n_1}$. But a_{n_2} is not a peak, so there is $n_3 > n_2$ with $a_{n_3} \geq a_{n_2}$. Continuing in this manner we get a subsequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$ that is monotone increasing. □

Theorem 24 (Bolzano-Weierstrass Theorem)**.** *If $\langle a_n \rangle_{n=1}^{\infty}$ is a bounded sequence, then it has a convergent subsequence.*

Proof. By the last lemma $\langle a_n \rangle_{n=1}^{\infty}$ has a monotone subsequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$. As the original sequence is bounded, any of its subsequences is also bounded. Thus $\langle a_{n_k} \rangle_{k=1}^{\infty}$ is a bounded monotone sequence and therefore it is convergent by Theorem 12. □

We proved the following in class.

Proposition 25. *Let S be a closed subset of* $\mathbf{R}$ *and $a = \lim_{n\to\infty} a_n$ where $a_n \in S$ for all n. Then $a \in S$.* □

We can now revisit some results form last semester.

Theorem 26. *If f is continuous on the closed bounded set S, then f is bounded on S. (That is there is a constant M such that $|f(x)| \leq B$ for all $x \in S$.)*

Proof. If this is false, then for positive each integer n there is a point $a_n \in S$ such that $|a_n| > n$. As S is a bounded set, the sequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$ is bounded and therefore by the Bolzano-Weierstrass Theorem it has a convergent subsequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$, say $\lim_{k\to\infty} a_{n_k} = a$. By Proposition 25 we have $a \in S$. Then by Theorem 9 $\lim_{k\to\infty} f(a_{n_k}) = f(a)$. Therefore by Theorem 6 $laf(a_{n_k}) \rangle_{k=1}^{\infty}$ is bounded so there is some constant B such that $|f(n_k)| \leq B$ for all k. But this implies $n_k < |f(a_{n_k})| \leq B$, which is impossible because $n_k \to \infty$ (as $n_k \geq n$). □

Theorem 27. *Any continuous function on a closed bounded set achieves both its maximum and minimum. That is if f is continuous on the closed bounded set S and there are points $a_{\max}$ and $a_{\min}$ such that $f(a_{\min}) \leq f(x) \leq f(a_{\max})$ for all $x \in S$.*

Problem 15*.* Prove this in the case of achieving the maximum. *Hint:* By the last theorem we know that f is bounded and therefore

$$A = \sup\{f(x) : x \in S\}$$

is finite. By definition of the supremum for each positive integer n there is a point $a_n \in S$ with

$$A - \frac{1}{n} < a_n \leq A.$$

This will have a convergent subsequence $\langle a_{n_k}\rangle_{k=1}^{\infty}$. Let $a_{\max} = \lim_{k\to\infty} a_{n_k}$ and show that $f(a_{\max}) = A$. $\square$

Problem 16. Compute the following limits. *Hint:* for this consider the mean value theorem.

(a) $\displaystyle\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$.

(b) $\displaystyle\lim_{n\to\infty} \left(\sqrt[3]{n^2+2n} - \sqrt[3]{n^2+n}\right)$.

Lemma 28 (Bernoulli's inequality)**.** *If $x > -1$ and n is a positive integer then*

$$(1+x)^n \geq 1 + nx.$$

When $n \geq 2$ the inequality is strict unless $x = 0$.

Proof. We did this last term as an example of proof by induction. $\square$

Problem 17. For positive integers n define

$$a_n := \left(1 + \frac{1}{n}\right)^n, \qquad b_n := \left(1 + \frac{1}{n}\right)^{n+1}.$$

Note

$$b_n = a_n\left(1 + \frac{1}{n}\right) > a_n.$$

(a) Prove $\langle a_n\rangle_{n=1}^{\infty}$ is an increasing sequence. *Hint:* First verify that

$$\frac{a_{n+1}}{a_n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1}\left(1 + \frac{1}{n}\right).$$

Now by Bernoulli's inequality with $x = -1/(n+1)^2$ we have

$$\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} > 1 + (n+1)\frac{-1}{(n+1)^2} = 1 - \frac{1}{n+1}.$$

Combine these facts to conclude

$$\frac{a_{n+1}}{a_n} > 1.$$

(b) Prove $\langle b_n\rangle_{n=1}^{\infty}$ is a decreasing sequence. *Hint:* This time look at the ratio

$$\frac{b_n}{b_{n+1}}$$

and do a variant on the last argument to show $\frac{b_n}{b_{n+1}} > 1$.

(c) Show both the limits

$$\lim_{n\to\infty} a_n \quad \text{and} \quad \lim_{n\to\infty} b_n$$

both exist and are equal to each other. *Hint:* Bounded monotone sequences are convergent. $\square$

This shows that the limit

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

exists. This was the original definition of e and came up in the problem of defining what it meant to compound interest continuously.